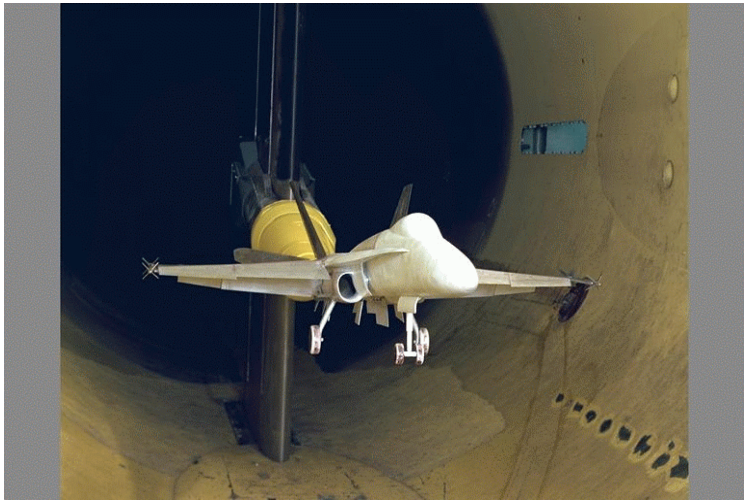
FE.1 Engineering Simulation

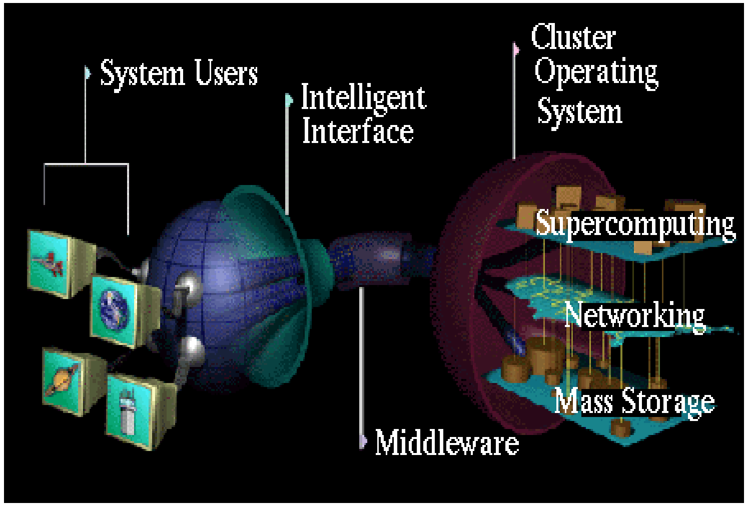
Physical Laboratory:

- ***model*** the geometry
  - similitude
  - cost
- ***measure*** the data
  - interpolation (errors)
  - ***interpretation***

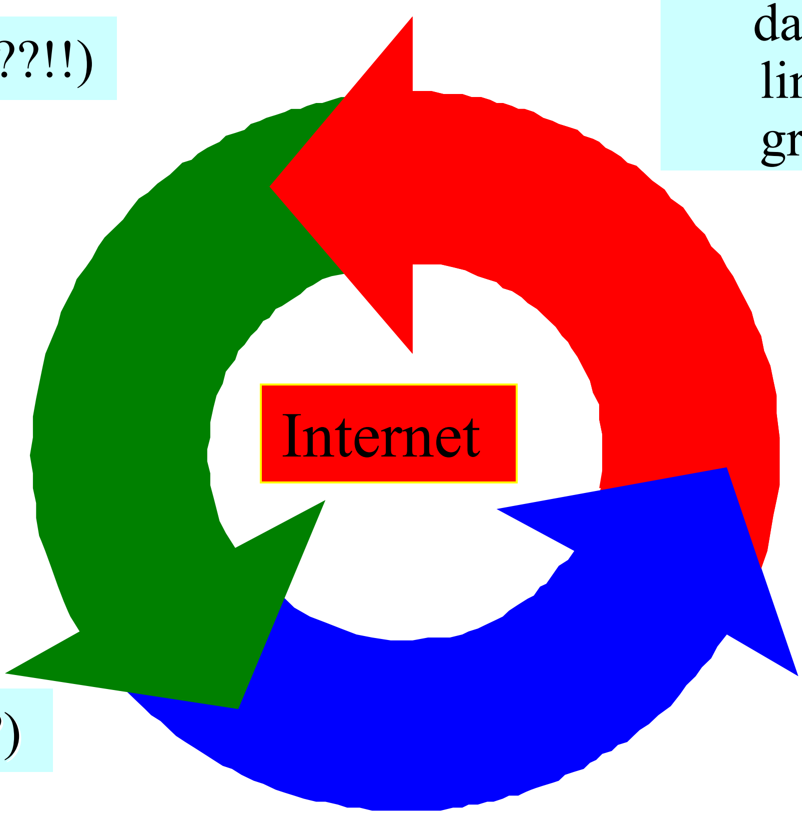


Computational (CFD) Laboratory:

- ***model*** the mathematics
  - conservation, BCs
- ***model*** the physics
  - complexity, cost
- ***compute*** the data
  - approximation error
  - physics model error
  - ***interpretation***



FE.2 A Problem Solving Environment



Computer Science

- computer platforms
- data management
- linear algebra
- graphics

knowledge (??!!)

Engineer

specified problem

Internet

knowledge (??)

knowledge (?)

Mathematics / Physics

- conservation principles
- physics closure models
- PDEs + BCs + IC, discrete methods

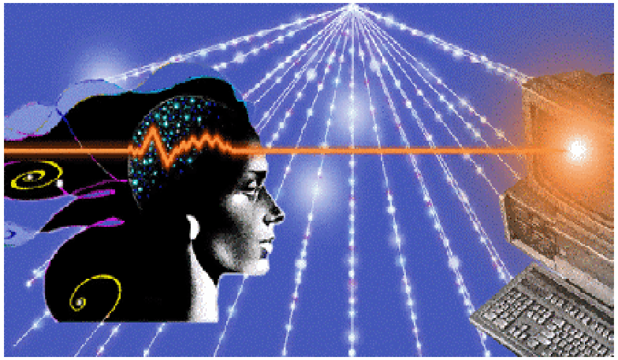
FE.3 Finite Element Computational Science

Collaboration:

engineering, mathematics, computer science

CFD Lab Problem Solving Environments:

Matlab, FEMLAB, *a*PSE, PICMSS



FE weak statement:

$$\mathrm{GWS}^h \Rightarrow \mathrm{S}_e\{\mathrm{WS}\}_e \equiv \{0\}$$

six basic objects to "program"

$$\{\mathrm{WS}\}_e \equiv \begin{pmatrix}\text{global}\\ \text{constant}\end{pmatrix}\begin{pmatrix}\text{element}\\ \text{average}\end{pmatrix}_e \begin{Bmatrix}\text{element}\\ \text{variables}\end{Bmatrix}_e \begin{pmatrix}\text{metric}\\ \text{data}\end{pmatrix}_e \begin{bmatrix}\text{FE master}\\ \text{matrix}\end{bmatrix}\begin{Bmatrix}\text{unknown}\\ \text{or data}\end{Bmatrix}_e$$

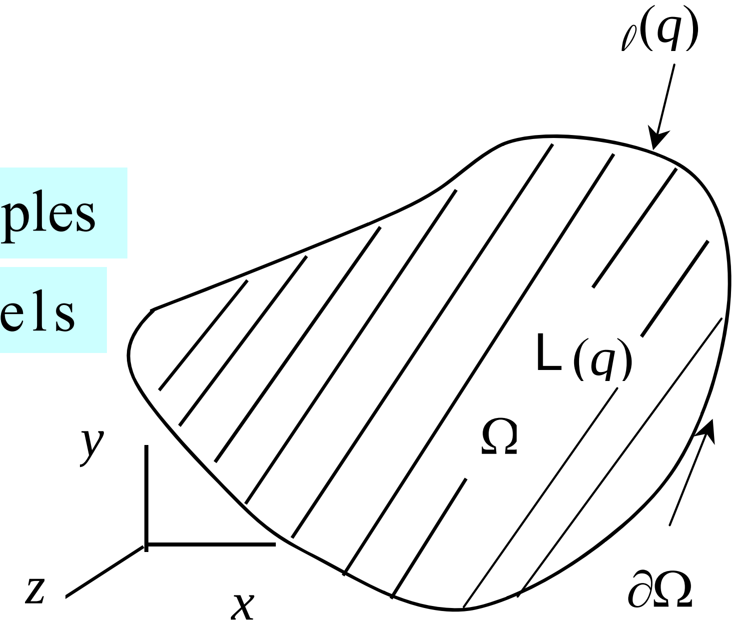
FE.4 Problem Statements in Engineering

Unknown (state variable) $q(\mathbf{x})$ satisfies a PDE

$$\mathsf{L}(q) = 0, \ \ on \ \ \Omega \subset \Re^n$$

e.g., mass, momentum, energy principles

+ physics closure models



Connection to specifics involves BCs

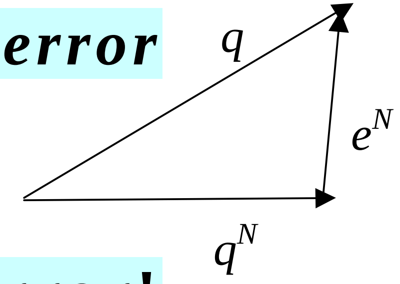
$$\ell(q) = 0, \ \ on \ \ \partial\Omega \subset \Re^{n\text{-}1}$$

Non-linearity, geometry preclude analytical solution

hence, seek an approximation

$$q(\mathbf{x}) \approx q^N(\mathbf{x}) = \sum_{\alpha}^{N} \Psi_\alpha(\mathbf{x})\, Q_\alpha$$

FE.5 WS Approximate Solution Process



Exact and approximate solutions differ by *error*

$$q(\mathbf{x}) = q^N(\mathbf{x}) + e^N(\mathbf{x})$$

Computational theory must *constrain* the error!

elegantly accomplished via a “weak statement”

$$\mathrm{WS} \equiv \int_\Omega \Phi_\beta(\mathbf{x})\,\mathrm{L}(q^N)\,\mathrm{d}\tau \equiv 0, \text{ for } \textit{all} \text{ test functions } \Phi_\beta$$

***Assuming* integrals can be completed, for $Q_\alpha \Rightarrow \{Q\}$**

$$\mathrm{WS} \Rightarrow [\ \mathrm{Matrix}\ ]\{Q\} = \{b\}$$

enforce BCs, solve for remaining unknowns in $\{Q\}$

FE.6 Decisions on Forming WS

Integrals forming WS require decisions on $\Phi(\mathbf{x})$ and $\Psi(\mathbf{x})$

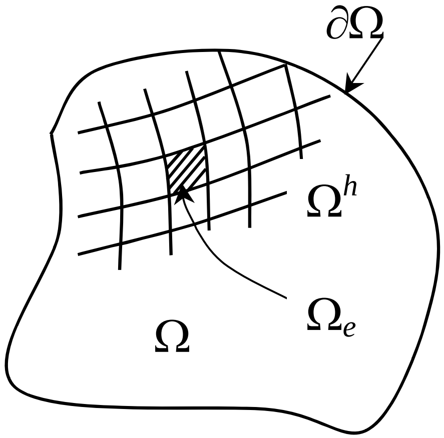
$\Phi(\mathbf{x}),\Psi(\mathbf{x})$	Example	WS Label
global	sine, cosine	analytical
	Chebyshev polynomials	spectral method
global-local	Chebyshev by blocks	pseudo-spectral
local	Lagrange polynomials	FE, FV, FD

Global functions – geometric limitations,
non-linear inflexibility

Local polynomial functions

employs domain + boundary *discretization*

$$\Omega \Rightarrow \Omega^h \equiv \cup_e \Omega_e$$



FE.7 Discrete Approximate Solution

Solution performance costs accrue to “meshing” Ω^h

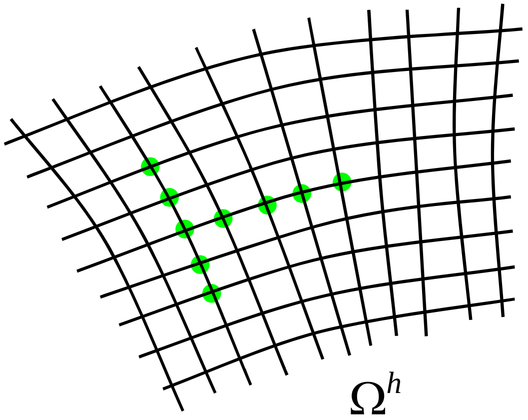
symbolically $q^N \equiv q^h = \cup_e q_e(\mathbf{x})$

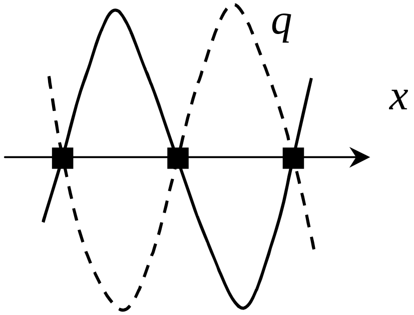
$\Rightarrow \{Q\}$

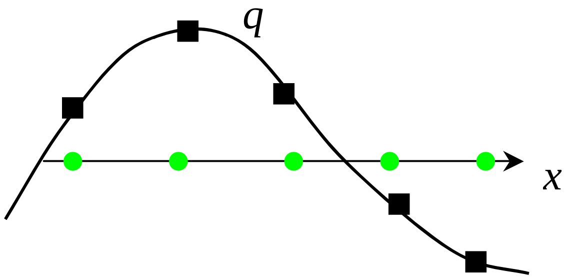
WS^h matrix *not* diagonal

$\Rightarrow$ linear algebra

spectral resolution compromised







■ $\Rightarrow \{Q\}$

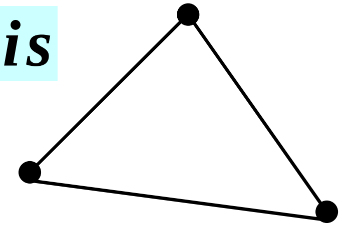
$2\Delta \boldsymbol{x}$ information is not resolved

$\Rightarrow \ \Omega^h$ requirements!

FE.8 Discrete Approximate Methods

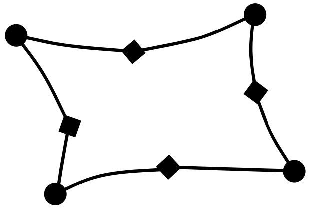
Finite elements employ trial space *basis*

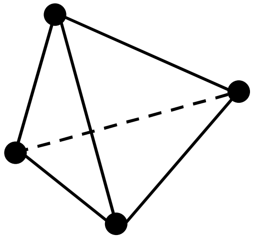
$$q^h = \cup_e q_e(\mathbf{x}), \quad and \quad q_e(\mathbf{x}) \equiv \{N_k(\eta)\}^T \{Q\}_e$$



Historical discrete methods predate FE

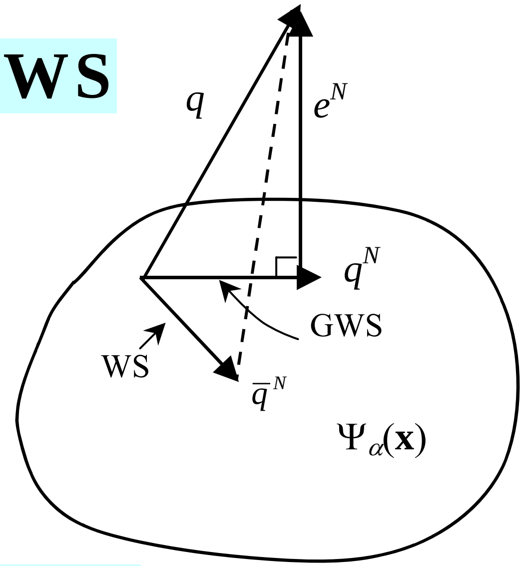
WS provides generalization





Name	**Trial, $\Psi(\mathbf{x})$**	**Test, $\Phi(\mathbf{x})$**
Galerkin (FE)	$\{N\}$	$\{N\}$
collocation	Lagrange polynomial	Kronecker δ
finite difference (FD)	not defined	not defined
finite volume (FV)	not defined	unity
least squares	$\{N\}$	$\mathcal{L}(\{N\})$
boundary element(BEM)	Lagrange polynomial	Green's function

FE.9 Optimal Weak Statement



Many choices exist for implementing WS

- *does an optimal selection for $\Psi_\alpha(\mathbf{x})$ and $\Phi_\alpha(\mathbf{x})$ discretized trial and test space basis sets exist?*

Engineer's choice is e^N minimum!

mathematicians can prove that

- *the discrete approximation error is minimized when the test and trial space basis functions are identical.*

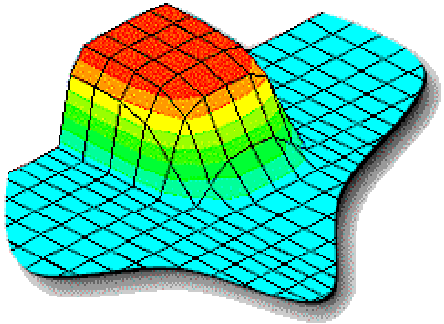
The FE implementation is the *Galerkin* WS^h

- GWS^h error is *orthogonal* to trial space basis set

$\Rightarrow$ *optimal !*

FE.10 Summary, FE Weak Statement

For arbitrary geometries and non-linearity



problem statement: $\mathcal{L}(q) = 0 \quad on \quad \Omega \subset \Re^n \quad + \quad \mathrm{BCs}$

approximation: $q(\mathbf{x}) \approx q^N(\mathbf{x}) \equiv \sum_{\alpha}^{N} \Psi_\alpha(\mathbf{x})\, Q_\alpha$

discretization: $\Omega \Rightarrow \Omega^h = \cup_e\, \Omega_e$

$q^N \equiv q^h = \cup_e\, \{N\}^T \{Q\}_e$

error extremization: $\mathrm{GWS}^N \equiv \int_\Omega \Psi_\beta \mathcal{L}(q^N)\, d\tau \equiv \{0\} \Rightarrow \mathrm{GWS}^h$

$\mathrm{GWS}^h = \mathrm{S}_e \{\mathrm{WS}\}_e \Rightarrow [\mathrm{Matrix}]\{Q\} - \{b\} = \{0\}$

linear algebra: $\{Q\} = [\ \mathrm{Matrix}\ ]^{-1}\{b\}$

error quantization: Ω^h refinements