IM.1 Introductory Material, PDE + BCs + IC

Laplacian dominates engineering PDE descriptions

due to "*physics model*" of *continuum mechanics*

yields EBV problem statements, hence BCs, IC

Example: unsteady heat conduction with source

D*E*: $$\frac{\partial T}{\partial t} = \kappa \nabla^2 T + s$$

steady, 1D form (ODE) $$\frac{\mathrm{d}^2 T}{\mathrm{d} x^2} + \frac{s}{\kappa} = 0 + \text{BCs}$$

Determine analytical solutions for:

BCs: $T(x_L) = T_L,\ T(x_R) = T_R$

sources: $s/\kappa \Rightarrow C,\ \sum_i C_i x^i,\ and\ \sum_n C_n \sin\left(\frac{n\pi x}{l}\right)$

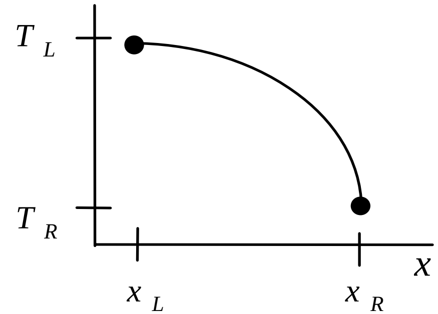
domain: $l \equiv x_R - x_L$

IM.2 Steady 1-D Conduction, Solutions

Constant source, integrate ODE twice

$$T(x) = \frac{1}{2} C x^2 + ax + b$$

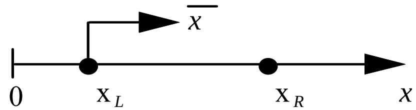
constants of integration are a and b
determine via BCs



Polynomial series source, integrate directly

$$T(x) = \frac{1}{(i+1)(i+2)} \sum_i C_i x^{i+2} + ax + b$$

translate origin, use BCs in $\bar{x}$ system



Fourier series source, use $\bar{x}$ system

trial space: $T(x) \equiv \sum_m A_m \sin\left(\frac{m\pi\bar{x}}{l}\right) + a\bar{x} + b$

substitute into ODE $\Rightarrow m, n, A_m, C_n$ + BCs (a, b)

IM.3 Unsteady 1-D Conduction Solution

Conclusion: 1-D linear *laplacian* ODE is easy to solve

Greater than 1-D employs *separation of variables*

find *trial space*

determine expansion coefficients using BCs and IC

Unsteady 1-D heat conduction, *s* = 0

PDE: $$\frac{1}{\kappa}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

BCs: $T(x=0,t) = 0 = T(x=l,t)$

IC: $T(x, t=0) = f(x)$

Trial space form via separation of variables (SOV)

$$T(x,t) = \sum_\alpha \Psi_\alpha(x,t) Q_\alpha \equiv F(x)G(t)$$

PDE becomes: $$\frac{1}{\kappa G}\frac{dG}{dt} = \frac{1}{F}\frac{d^2F}{dx^2} \equiv \pm\beta^2$$

IM.4 Unsteady 1-D Conduction Solution

Separation of variables: PDE $\Rightarrow$ 2 ODEs

functional *independence* requires the constant β^2

ODEs now have source term functional dependence!

ODE(x): $$\frac{d^2F}{dx^2} \mp \beta^2 F = 0$$

ODE(t): $$\frac{dG}{dt} = \pm\beta^2 \kappa G$$

Must choose sign for β^2 to maintain solution boundedness

solutions: $F(x) = A \sin(\beta x) + B \cos(\beta x)$

$$G(t) = C \exp(-\kappa\beta^2 t)$$

IM.5 Separation of Variables, BCs

Combining *F* and *G* produces trial space members

$$\Psi_\alpha(x, t) Q_\alpha = \left[A_\alpha \sin(\beta x) + B_\alpha \cos(\beta x)\right] \exp\left(-\kappa\beta^2 t\right)$$

Contains 3 arbitrary coefficients: A_α, B_α and β^2

BCs:

$$A_\alpha \sin(0) + B_\alpha \cos(0) = 0$$

$$A_\alpha \sin(\beta l) + B_\alpha \cos(\beta l) = 0$$

non-trivial solution:

$$B_\alpha = 0 \textit{ and } \beta \Rightarrow \beta_n = n\pi / l, \qquad n = 0, 1, 2, \ldots$$

Switching index label, solution *trial space* member is

$$\Psi_n(x, t) = \sin\left(\frac{n\pi x}{l}\right) \exp\left(-\kappa\beta_n^2 t\right)$$

IM.6 Separation of Variables, GWS

Trial space basis obtained via SOV process

series solution must satisfy IC, hence at $t = 0$

IC:

$$T(x, t=0) \equiv f(x) = \sum_{n}^{\infty} Q_n \sin\left(\frac{n\pi x}{l}\right) \exp(0)$$

one equation for *infinite* number of unknowns Q_n

Matrix solution via *optimal* Galerkin weak statement

$$\text{GWS} \equiv \int_{\Omega} \sin\left(\frac{m\pi x}{l}\right) f(x)\mathrm{d}x - \sum_{n}^{\infty} \sin\left(\frac{m\pi x}{l}\right) Q_n \sin\left(\frac{n\pi x}{l}\right)\mathrm{d}x \equiv 0$$

for $m = 1, 2, \ldots, \mathrm{M}, \ldots$

Since sin(*n*, *m*) are *orthogonal*, matrix is diagonal

$$\Rightarrow Q_n \equiv \frac{2}{l}\int_0^l \sin\left(\frac{n\pi x}{l}\right) f(x)\,\mathrm{d}x$$

IM.7 Theoretical Foundations, GWS

Analytical SOV solution examples, ODEs, PDEs

- *identify the trial space for the approximation,*
- *form the expansion on* Q_α*,*
- *determine optimal coefficients by evaluating the* GWS
- however, SOV infinite series PDE solution intractable !

Resolution, truncate series for an *approximation*

$$T(x,t) \cong T^N(x,t) \equiv \sum_n^N Q_n \sin\left(\frac{n\pi x}{l}\right) \exp\left(-\kappa\beta_n^2 t\right)$$

sin (n) belongs to a *complete* set

select N for verifiable *accuracy*

Analagously, GWS^h discrete implementation via FE on Ω^h

asymptotic error estimate

mesh *refinement* $\Leftrightarrow$ N increasing

IM.8 Steady 3-D Conduction SOV Solution

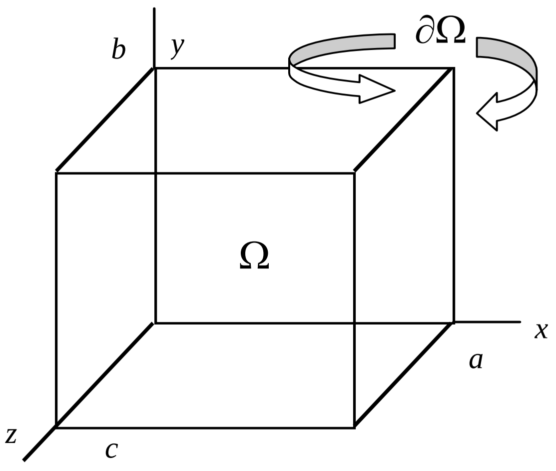
Steady conduction on 3-D cube domain

$$\text{PDE}\ : \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

$$\text{BCs}\ : T(0, y, z) = 0 = T(a, y, z)$$

$$T(x, 0, z) = 0 = T(x, b, z)$$

$$T(x, y, 0) = 0, \textit{ and } T(x, y, c) = f(x, y)$$



Trial space determination

form for SOV: $T(x, y, z) = X(x)\, Y(y)\, Z(z)$

homogeneous BCs: $\Psi_{n,m}(\mathbf{x}) \equiv \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sinh\left(\pi\beta_{n,m} z\right)$

GWS on coefficient matching $f \Rightarrow$ diagonal matrix

$$Q_{n,m} \equiv \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \mathrm{d}x\, \mathrm{d}y$$

approximation: $T^N(\mathbf{x}) \equiv \sum_{n,m}^{N=M} \Psi_{n,m}(\mathbf{x})\, Q_{n,m}$

IM.9 Summary, Analytical PDE Methods

Restrictions for analytic SOV "to work"

- PDE linear
- BCs separable on $\partial\Omega$
- quasi-linear data

Attribute assimilations with weak statement

- solution trial space
- orthogonality
- completeness
- predictable accuracy enhancement

Finite element implementation of GWS handles

- PDE non-linear systems
- BCs on arbitrary $\partial\Omega$
- arbitrary data, first derivatives

IM.10 Legacy FD Connections To WS

Finite difference methodology

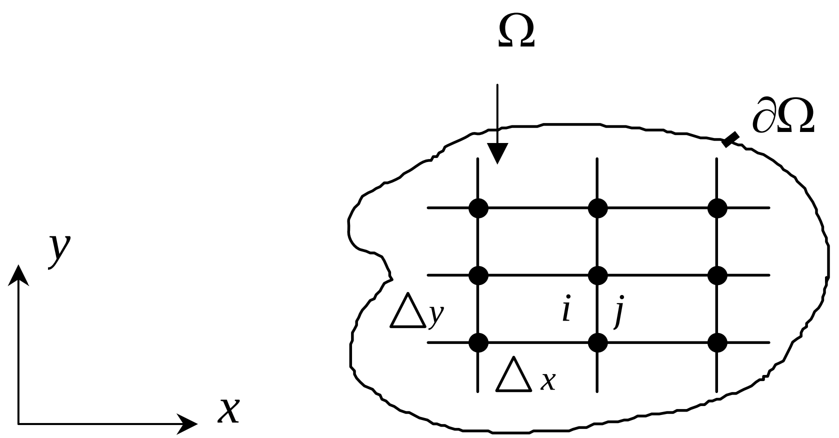
Taylor series for PDE operators

essentially a 1-D process

Example problem:

$$\mathrm{L}(\phi) = \nabla^2\phi + s = 0$$

$$\ell(\phi) \Rightarrow \phi = \text{constant}$$



For cartesian interior meshing with measures Δx, Δy

$$\phi(x+\Delta x, y) = \phi(x, y) + \Delta x \left.\frac{\partial\phi}{\partial x}\right|_{x,y} + \frac{1}{2}\Delta x^2 \left.\frac{\partial^2\phi}{\partial x^2}\right|_{x,y} + O(\Delta x^3)$$

$$\Rightarrow \left.\frac{\partial\phi}{\partial x}\right|_{x,y} \equiv \left.\frac{\partial\phi}{\partial x}\right|_{i,j} = \frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x} + O(\Delta x)$$

IM.11 FD Order-Of-Accuracy Control

Backwards TS yields $O(\Delta x, \Delta y)$ constructions

$$\phi(x-\Delta x, y) = \phi(x, y) - \Delta x \left.\frac{\partial \phi}{\partial x}\right|_{x,y} + \Delta x^2 \left.\frac{\partial^2 \phi}{\partial x^2}\right|_{x,y} + $$

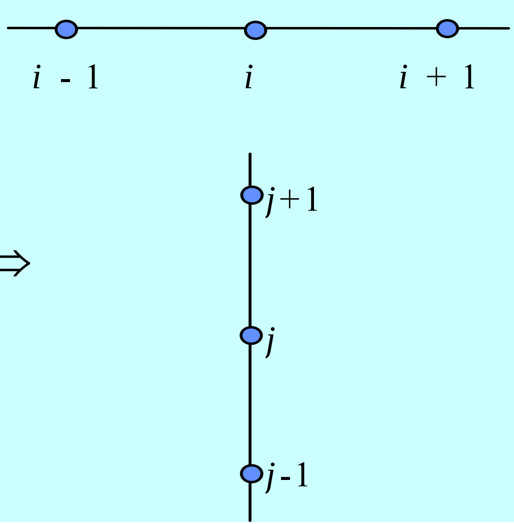
$$\Rightarrow \left.\frac{\partial \phi}{\partial x}\right|_{i,j} = \frac{\phi_{i,j} - \phi_{i-1,j}}{\Delta x} + O(\Delta x)$$

Subtracting TSs $\Rightarrow O(\Delta x^2, \Delta y^2)$ "molecules"

$$\left.\frac{\partial \phi}{\partial x}\right|_{i,j} = \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta x} + O(\Delta x^2)$$

$$\left.\frac{\partial \phi}{\partial y}\right|_{i,j} = \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2\Delta y} + O(\Delta y^2)$$

$\Rightarrow$



IM.12 FD Stencils, Laplacian

FD operators written as *stencils*

$$\left.\frac{\partial \phi}{\partial x}\right|_{i,j} = \frac{1}{2h}\begin{pmatrix} -1 & 0 & 1 \end{pmatrix} + O(h^2), \qquad \left.\frac{\partial \phi}{\partial y}\right|_{i,j} = \frac{1}{2k}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + O(k^2)$$

using h, k, to replace Δx, Δy

Via same processes 2nd derivative FD stencils

$$\left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j} = \frac{1}{h^2}\begin{pmatrix} 1 & -2 & 1 \end{pmatrix} + O(h^2), \qquad \left.\frac{\partial^2 \phi}{\partial y^2}\right|_{i,j} = \frac{1}{k^2}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + O(k^2)$$

Hence, 2-D laplacian FD stencil (“symbol”) for *h* = *k*

$$\left.\nabla^2\right|_{i,j} = \frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} + O(h^2)$$

3-D laplacian FD symbol is the obvious extension

IM.13 FD Symbol ⇔ GWS Matrix Statement

FD symbol for laplacian PDE on uniform mesh

$$\left(\nabla^2\phi + s\right)_{\mathrm{FD}} \Rightarrow \frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix}\Phi_{i,j} + S_{i,j} + \mathrm{TE} = 0$$

a "geometric" picture of matrix statement

Symbol correspondence to GWS statement

$$\frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix}\Phi_{i,j} + S_{i,j} + \mathrm{TE} \Rightarrow [\mathrm{Matrix}]\{\Phi\} - \{b(s)\} + \{\mathrm{error}\} = \{0\}$$

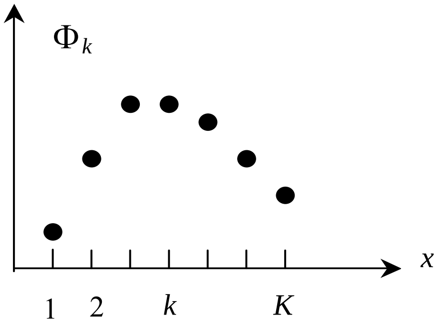
GWS^h generates matrix statement via FE *assembly*

FD stencil ⇔ FE [Matrix] connections will become exposed

IM.14 FD Approximate Solution

FD process yields a set of *numbers* Φ_k, $1 \le k \le$ nnode

FD numbers $\Leftrightarrow$ approximate solution employs interpolation concepts



Interpolation polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{\alpha=0}^{N} a_\alpha x^\alpha$$

$N+1$ coefficients a_α produce N^{th}-degree polynomial

Using nodal data, an FD "solution" could be

$$f(x_k) = \Phi_k = a_0 + a_1 x_k + a_2 x_k^2 + \ldots + a_N x_k^N \quad \text{for } 1 \le k \le K$$

$\Rightarrow$ [Matrix] statement for determining the a_α

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^N \\ 1 & x_2 & \cdots & & & \\ \vdots & & & & & \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{Bmatrix} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_N \end{Bmatrix}$$

IM.15 Polynomial FD "Solution"

Evaluating polynomial at K FD nodes x_k produces

$$[\text{ Matrix } (x_k^{\alpha})] \{ a_k \} = \{ \Phi_k \}$$

solvability requires $K = N + 1$

involves α-exponentiation on x_k

Solution of [Matrix] statement, e.g., via Cramer's rule

FD solution: $$\Phi\big|_{FD} \Rightarrow f(x) = \sum_{\substack{\alpha=0,k-1 \\ k=1,K}} a_{\alpha}\left(x_k, \Phi_k\right) x^{\alpha}$$

detraction is oscillations between nodes for $K \geq 6$

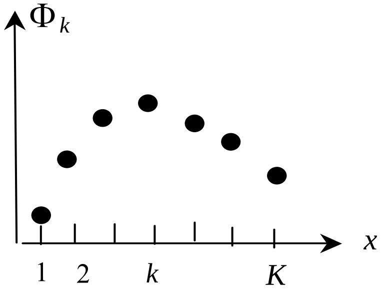
resolution is *piecewise continuous* Lagrange polynomials

IM.16 Lagrange Polynomial Interpolation

Functionally convenient interpolation polynomial

Lagrange: $f(x) \equiv \sum_{k=1}^{K} L_K(x, x_k)\,\Phi_k$

rather than: $f(x) = \sum_{\alpha} a_\alpha(x_k, \Phi_k)\, x^\alpha$



Lagrange polynomial, for *K* data points

$$f(x_j) \equiv \Phi_j = \sum_{k=1}^{K} L_K(x_j, x_k)\,\Phi_k = \delta_{jk}\,\Phi_k$$

for the K-1 zeros of δ_{jk}

$$L_K(x, x_k) = C_k\,(x - x_1)\,(x - x_2)\ \ldots\ (x - x_{k-1})\,(x - x_{k+1})\ \ldots\ (x - x_K)$$

for δ_{jk} at unity:

$$C_k = 1\,/\,(x_k - x_1)\,(x_k - x_2)\ \ldots\ (x_k - x_{k-1})\,(x_k - x_{k+1})\ \ldots\ (x_k - x_K)$$

IM.17 Lagrange Interpolation Polynominals

Lagrange polynomial of degree K-1

$$L_K(x, x_k) = \frac{(x - x_1)\,(x - x_2)\,\ldots\,(x - x_{k-1})\,(x - x_{k+1})\,\ldots\,(x - x_K)}{(x_k - x_1)\,(x_k - x_2)\,\ldots\,(x_k - x_{k-1})\,(x_k - x_{k+1})\,\ldots\,(x_k - x_K)}$$

polynomial interpolation for FD data Φ_k

$$\phi(x)_{\mathrm{FD}} = \sum_{k=1}^{K} L_K(x, x_k)\,\Phi_k(\mathrm{FD})$$

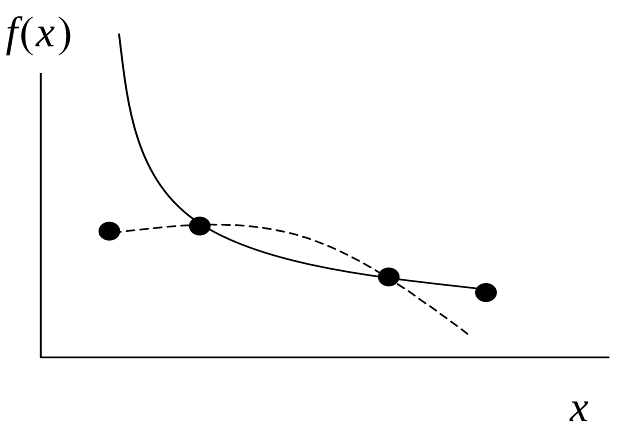
provides a conceptual connection to GWS starting point

$$q^N(x) = \sum_{\alpha=1}^{N} \Psi_\alpha(x) Q_\alpha$$

Piecewise continuity issues

depends on K

not unique for $K > 1$



IM.18 GWS Summary, FE Trial Space Basis

FE global solution approximation, ∪ denotes “union”

$$q^N \equiv q^h = \cup_e q_e(x), \; and \quad q_e(x) \equiv \{N_k(\eta)\}^T \{Q\}_e$$

key is FE trial space basis $\{N k(\eta)\}$

construction via Lagrange polynomials extends directly to

$k > 1$
$n > 1$

involves local (intrinsic) coordinate system η

coordinate transformations $\boldsymbol{x} = \boldsymbol{x}(\eta)$

Solving GWS determines expansion coefficient set

[Matrix] solution $\Rightarrow$ $\{Q\} = \cup_e \{Q\}_e$