PS.1 Engineering Simulation

Fundamentally, engineers seek "*solutions*" to "problems"

conservation principles

constitutive closure ("physics models")

$\Rightarrow$ vector differential calculus

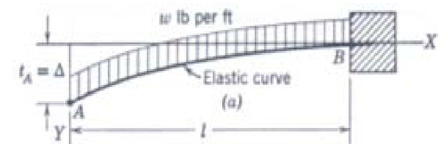
Conservation principles, *Lagrangian* viewpoint

mass:	$\mathrm{d}M = 0, \quad M = \Sigma\, m_i$
linear momentum:	$\mathrm{d}\mathbf{P} = \Sigma\,\mathbf{F}, \quad \mathbf{P} = M\,\mathbf{V}$
energy:	$\mathrm{d}E = \mathrm{d}Q - \mathrm{d}W$
thermodynamic process:	$\mathrm{d}S \geq 0$

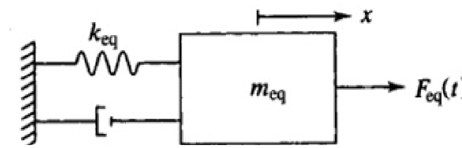
PS.2 Continuum Mechanics Viewpoint

Lagrangian form illustrative applications

strength of materials: $d^2y/dx^2 = M/EI$



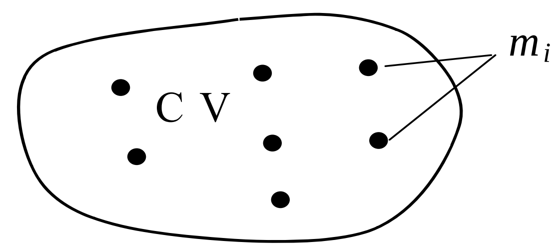
mechanical vibrations: $m\ddot{x} + c\dot{x} + kx = F(t)$



rigid body dynamics: $\mathbf{a}_{XYZ} = \mathbf{a}_{xyz} + \ddot{\mathbf{R}} + 2\boldsymbol{\omega} \times \mathbf{V}_{xyz} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \boldsymbol{\rho}$

Continuum mechanics concept

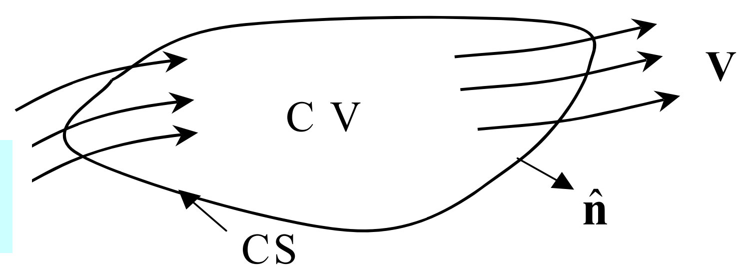
$$\rho(\mathbf{x},t) = \lim_{CV \to 0} \frac{1}{CV} \sum_i m_i$$



Control volume (CV) with enclosing surface (CS)

Reynolds transport theorem

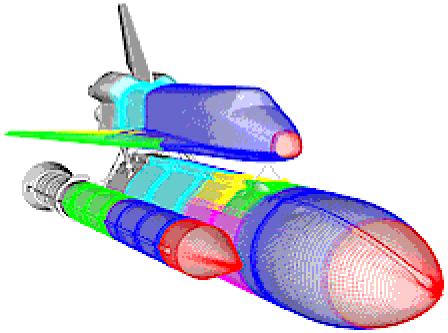
$$d(\ ) \Rightarrow D(\ ) \equiv \frac{\partial}{\partial t}\int_{CV} (\ )\, d\tau + \oint_{CS} (\ )\, \mathbf{V} \cdot \hat{\mathbf{n}}\, d\sigma$$



PS.3 Continuum Mechanics Principles

Continuum descriptions using $\rho(\mathbf{x}, t)$

structures | electromagnetics
fluids | mass transport
heat transfer | mechanical vibrations



Conservation principles, *Eulerian* viewpoint

$$\mathrm{D}M \Rightarrow \frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho \mathrm{d}\tau + \oint_{\mathrm{CS}} \rho \mathbf{V} \cdot \hat{\mathbf{n}} \mathrm{d}\sigma = 0$$

$$\mathrm{D}\mathbf{P} \Rightarrow \frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho \mathbf{V} \mathrm{d}\tau + \oint_{\mathrm{CS}} \mathbf{V} \rho \mathbf{V} \cdot \hat{\mathbf{n}} \mathrm{d}\sigma = \int_{\mathrm{CV}} \rho \mathbf{B} \mathrm{d}\tau + \int_{\mathrm{CS}} \mathbf{T} \mathrm{d}\sigma$$

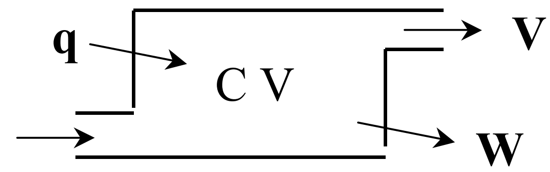
$$\mathrm{D}E \Rightarrow \frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho e \mathrm{d}\tau + \oint_{\mathrm{CS}} (e + p/\rho) \rho \mathbf{V} \cdot \hat{\mathbf{n}} \mathrm{d}\sigma = \int_{\mathrm{CV}} s \mathrm{d}\tau + \oint_{\mathrm{CS}} (\mathrm{W} - \mathbf{q} \cdot \hat{\mathbf{n}}) \mathrm{d}\sigma$$

PS.4 Continuum Mechanics Forms

Control volume Reynolds form rarely used

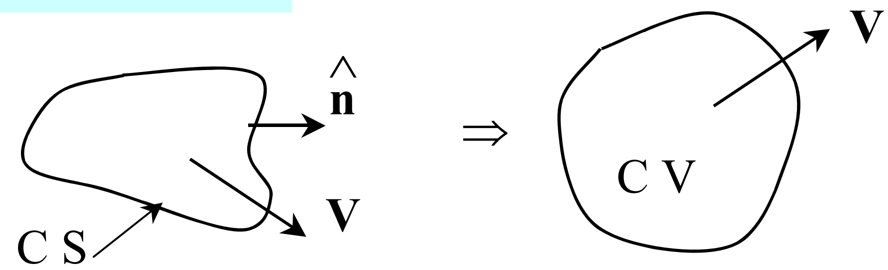
"network" simulations

"coarsest mesh" possible



For stationary CV, *Divergence Theorem*

$$\oint_{CS} \rho \mathbf{V} \cdot \hat{\mathbf{n}} d\sigma = \int_{CV} \nabla \cdot \rho \mathbf{V} d\tau$$

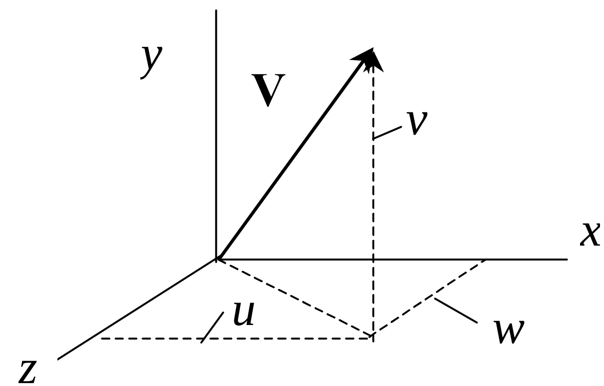


gradient **vector** derivative

$$\nabla = \hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z}$$

divergence operation

$$\nabla \cdot \rho \mathbf{V} = \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}$$

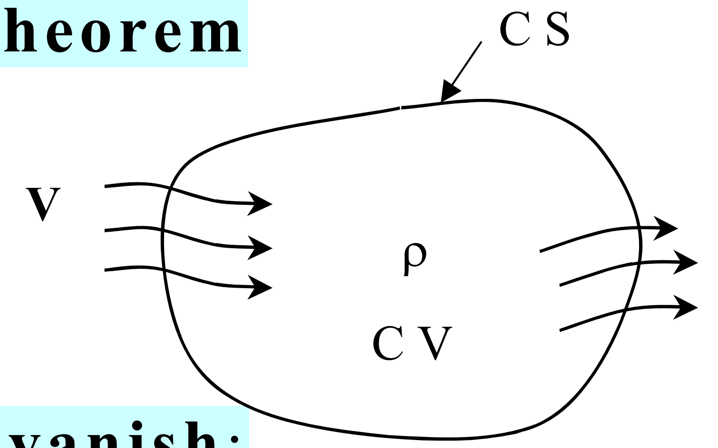


vectors!! calculus!!

PS.5 Continuum Mechanics PDEs

Reynolds transport + Divergence theorem

$$DM \quad \Rightarrow \int_{CV} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} \right) d\tau = 0$$



For arbitrary CV, integrand must vanish:

$$DM : \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0$$

$$D\mathbf{P} : \frac{\partial \rho \mathbf{V}}{\partial t} + \nabla \cdot \rho \mathbf{V}\mathbf{V} = \rho \mathbf{g} + \nabla \mathrm{T}$$

$$DE : \frac{\partial \rho e}{\partial t} + \nabla \cdot (\rho e + p)\mathbf{V} = s - \nabla \cdot \mathbf{q}$$

for traction vector **T**, gravity body force **g**, heat flux **q**

PS.6 Physics Closure Models

Continuum PDEs universally valid!

solids, fluids, EM, vibrations $\Rightarrow$ *how*?!

Distinction for discipline $\rightarrow$ *physics closure models*

example, Fourier *conduction law*: $\mathbf{q} \equiv -k\nabla T$

k = thermal conductivity $\Rightarrow k(\mathbf{x}, T)$

- sign for heat flow direction

Unsteady heat conduction: for $\mathbf{V} = \mathbf{0}$, $e = c_V T$

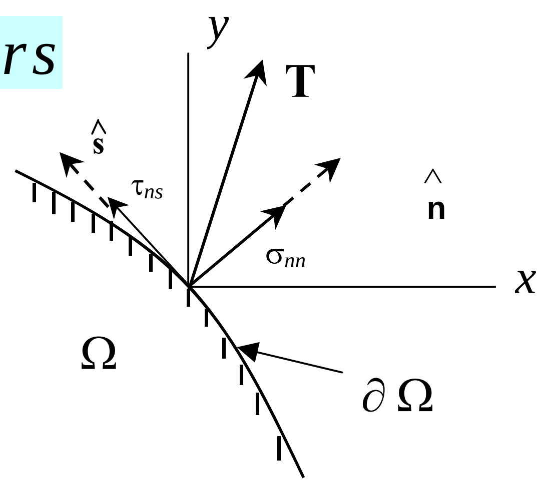
DE: $\dfrac{\partial T}{\partial t} = \kappa \nabla^2 T + s$

$\kappa = k/\rho c_p \Rightarrow$ material *thermal diffusivity*

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ = Laplace PDE operator

PS.7 Physics Closure for DP, Structures

Structural mechanics: DP involves *tensors*



traction: $\mathbf{T}_{cs} \Rightarrow \tau_{ij}\hat{\mathbf{n}}_j$

Statics: $\partial/\partial t = 0 = \mathbf{V}$

$$\mathrm{DP}: \quad \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g_x = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho g_y = 0$$

closure model: *linear* Hooke's law, $n = 2$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\upsilon^2} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & (1-\upsilon)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

kinematics (strain-displacement): $\varepsilon_x = \dfrac{\partial u}{\partial x}, \quad \gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$

$$\mathrm{DP}: \left.\begin{aligned} \nabla^2 u - \frac{1}{1-2\upsilon}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\rho g_x}{G} = 0 \\ \nabla^2 v - \frac{1}{1-2\upsilon}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\rho g_y}{G} = 0 \end{aligned}\right\} \text{ laplacian PDE system}$$

PS.8 Structural Mechanics D*E* Principle

Structural analyses employ an *energy principle*

$$\mathrm{D}E: \qquad \Pi \equiv \int_{\Omega} \mathrm{d}e = \frac{1}{2}\int_{\Omega} \mathrm{d}vol \int_{0}^{\varepsilon} \tau_{ij}\,\mathrm{d}\varepsilon_{ij} - \int_{\partial\Omega} u_j T_j \,\mathrm{d}surf$$

Principle of Virtual Work: inserting Hooke's law

$$\mathrm{D}E: \qquad \Pi = \int_{\Omega}\left(\frac{1}{2}\{\varepsilon\}^{T}[\mathbf{E}]\{\varepsilon\} - \{\mathbf{u}\}^{T}\{\mathbf{B}\}\right)\mathrm{d}vol - \int_{\partial\Omega}\{\mathbf{u}\}^{T}\{\mathbf{T}_s\}\mathrm{d}surf$$

Extremum of D*E* ⇒ DP, e.g.,

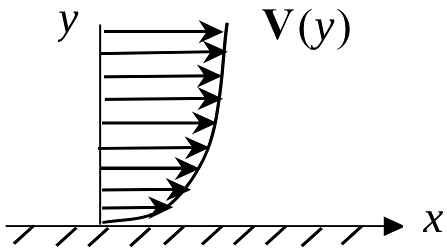
$$\mathrm{DP:} \quad L(\mathbf{u}) = -\nabla^2\mathbf{u} - g(\upsilon)\,\nabla(\nabla\bullet\mathbf{u}) - \mathbf{b} = 0$$

PS.9 Physics Closure for Fluids

Fluid mechanics: strain $\Rightarrow$ *strain-rate*, hence velocity $\mathbf{V} \Rightarrow u_i$

viscosity closure model: Stoke's law

$$\tau_{ij} = \upsilon \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] - \frac{2}{3} \lambda \nabla \cdot \mathbf{V} \delta_{ij}$$



Navier-Stokes PDE system: incompressible, 2D, steady

D*M* : $\nabla \cdot \mathbf{V} = 0$

D**P** :

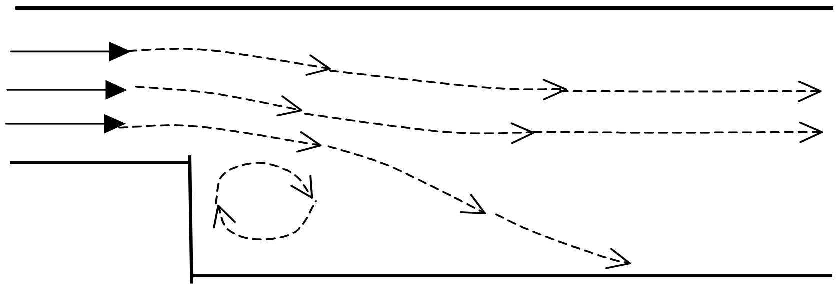
$$\upsilon \nabla^2 u - \frac{1}{\rho_o} \frac{\partial p}{\partial x} - \frac{\partial uu}{\partial x} - \frac{\partial uv}{\partial y} = 0$$

$$\upsilon \nabla^2 v - \frac{1}{\rho_o} \frac{\partial p}{\partial x} - \frac{\partial uv}{\partial x} - \frac{\partial vv}{\partial y} = 0$$

$\Rightarrow$ laplacian PDE system

non-linear(!)

constrained

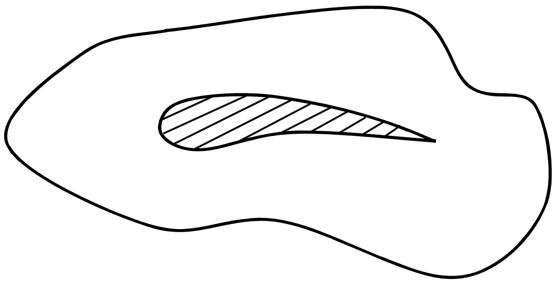


PS.10 Continuum Laplacian PDE Systems

Fluid mechanics: irrotational $(\nabla \times \mathbf{V} = 0)$
\+ incompressible $(\nabla \cdot \mathbf{V} = 0)$

D *M*: $\nabla^2 \Phi = 0\,, \quad \mathbf{V} = - \nabla \Phi$

potential theory



Wave propagation: Maxwell's equations, plane wave

D*M*: $\nabla^2 \phi - \omega^2 \phi = 0$

$\Rightarrow$ perfect medium, ϕ = volts, ω = frequency

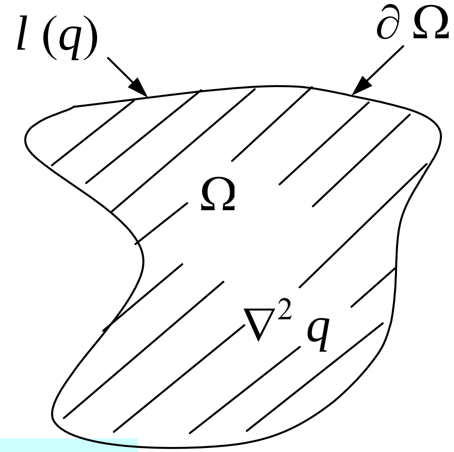
Mass transport: conservation of species (*implicitly* non-linear)

D*M*: $\nabla^2 \phi + s(\phi) = 0$

Creeping flow: saturated aquifer (*explicitly* non-linear)

D*M*: $\nabla \cdot \phi \nabla \phi = 0$

PS.11 Engineering Problem Statements



Summary: problem statements are PDEs!

Laplace operator (∇^2) is highest derivative
$\Rightarrow$ physics closure model

with sources, non-linearities, unsteady terms

Generic *elliptic* boundary value (EBV) problem

$$\mathrm{L}(q) \;=\; -\nabla^2 q - s\,(q) = 0, \quad on \quad \Omega$$

boundary conditions required on total $\partial\Omega$

$$l\,(q) \;=\; \nabla q \cdot \widehat{\mathbf{n}} + g(q) = 0 \;\; on \;\; \partial\Omega$$

q	$=$ constant	(Dirichlet)
$\nabla q \cdot \widehat{\mathbf{n}}$	$=$ fixed	(Neumann)
$\nabla q \cdot \widehat{\mathbf{n}}$	$= -g(q)$	(Robin)

PS.12 Computational Simulation

Engineering design problems: PDEs + physics + BCs

unknown called *state variable* ≡ $q\ (\mathbf{x},\ t)$

solution is distribution of q on $(\mathbf{x},\ t > t_0)$

analytically *intractible*!

Computer simulation ⇒ seek an *approximate* solution

$$q(\mathbf{x},t) \approx q^N(\mathbf{x},t)$$

finite difference - historical, archaic

finite element analysis

$$q^N(\mathbf{x},t) \equiv q^h(\mathbf{x},t) = \cup_e q_e(\mathbf{x},t)$$

optimal
encompassing
real world problems