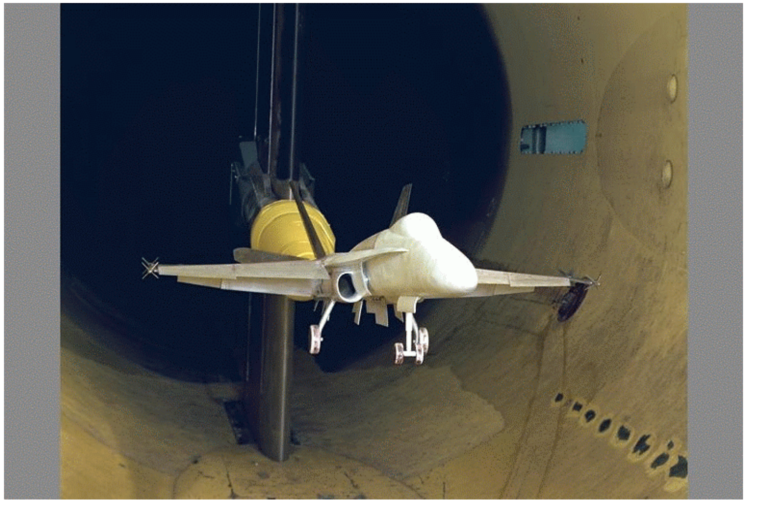
S.1(FE.1) Engineering Simulation

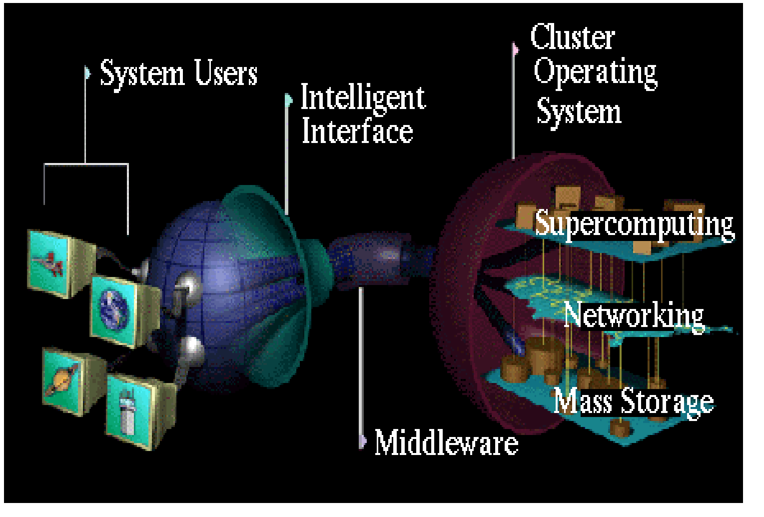
Physical Laboratory:

- ***model*** the geometry
  similitude
  cost
- ***measure*** the data
  interpolation (errors)
  interpretation



Computational Laboratory:

- ***model*** the mathematics
  conservation, BCs
- ***model*** the physics
  complexity, cost
- ***compute*** the data
  approximation error
  physics model error
  interpretation



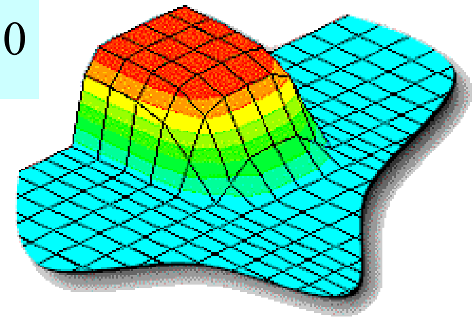
S.2(FE.10) Summary, Finite Element Analysis

For arbitrary geometries and non-linearity

problem statement: $\mathcal{L}(q) = 0 \quad on \quad \Omega \subset \Re^n \quad + \quad \text{BCs}$

approximation: $q(\mathbf{x}) \approx q^N(\mathbf{x}) \equiv \sum_{\alpha}^{N} \Psi_\alpha(\mathbf{x})\, Q_\alpha$

error minimization: $\text{GWS}^N = \int_{\Omega_e} \Psi_\alpha(\mathbf{x})\, \mathcal{L}(q^N)\, d\tau \equiv 0$



FE discretization: $\Omega \approx \Omega^h = \cup_e \Omega_e$

$q^N \equiv q^h = \cup_e \{N(\mathbf{x})\}^T \{Q\}_e$

FE GWS^h: $[\text{ Matrix }]\{Q\} = \{b\}$

error quantization: refined Ω^h solutions

S.3(PS.12) Computational Simulation

Engineering design problems: PDEs + physics + BCs

unknown called *state variable* ≡ $q\ (\mathbf{x}, t)$

solution is distribution of q on $(\mathbf{x}, t > t_0)$

analytically *intractible*!

Computer simulation ⇒ seek an *approximate* solution

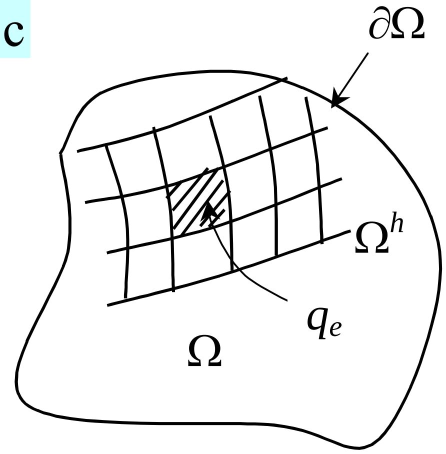
$$q(\mathbf{x},t) \approx q^N(\mathbf{x},t)$$

finite difference - historical, archaic

finite element analysis

$$q^N(\mathbf{x},t) \equiv q^h(\mathbf{x},t) = \cup_e q_e(\mathbf{x},t)$$

optimal
encompassing
real world problems



S.4(HT1.18) Error Estimation, Energy Norm

Improved error estimate uses entire solution via a "norm"

$$\textit{energy norm} \equiv \left\| T \right\|_E^h \equiv \frac{1}{2}\int_\Omega k \frac{dT^h}{dx}\frac{dT^h}{dx} d\tau \Rightarrow \frac{1}{2}\sum_e^{M} \{Q\}_e^T [\text{DIFF}]_e \{Q\}_e$$

Uniform mesh refinement study

$$\left\| T^h \right\|_E + \left\| e^h \right\|_E = \left\| T \right\|_E = \left\| T^{h/2} \right\|_E + \left\| e^{h/2} \right\|_E = \ldots$$

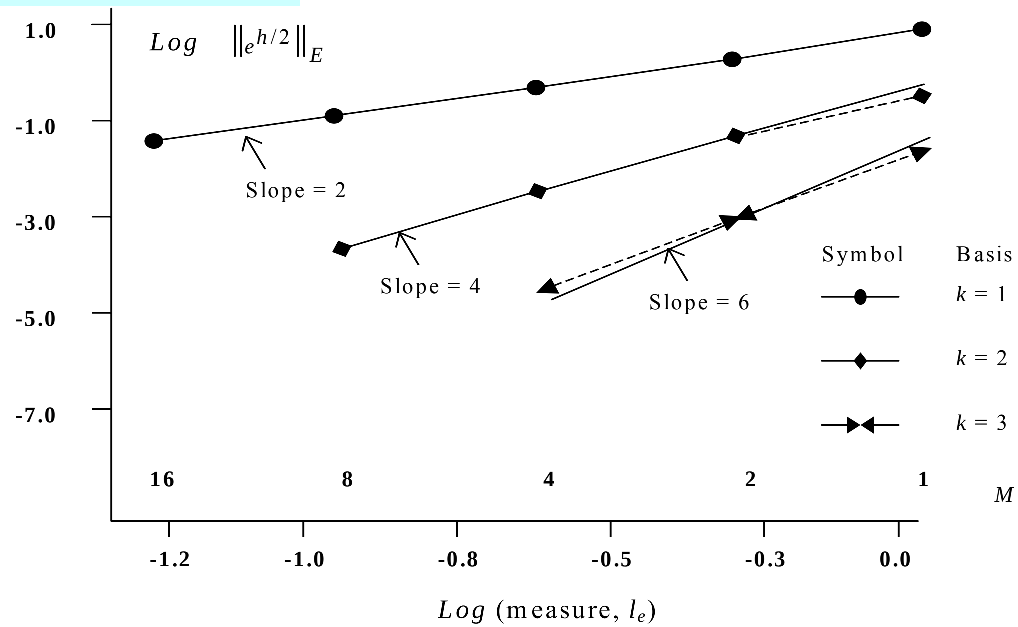
$$\textit{asymptotic convergence}: \left\| e^h \right\|_E \le C_k \ell_e^{2k} \left\| \text{data} \right\|_{L2}^2$$

error estimator

$$\| e^{h/2} \|_E = \frac{\Delta \| T^{h/2} \|_E}{2^{2k} - 1}$$

confirmation of theory

$$\textit{slope} = \frac{\log \| e^{h/M} \|_E / \| e^{h/2M} \|_E}{\log 2}$$



S.5(HT1.28) DE GWS^h Summary, n = 1

Given DE + BC problem statement on n = 1

$$\mathcal{L}(q) = 0 \; on \; \Omega \subset \Re^1, \qquad \ell(q) = 0 \quad on \; \partial\Omega$$

FE weak statement recipe

approximation: $T(x) \approx T^N(x) \equiv T^h(x) = \cup_e T_e(x)$

FE basis: $T_e(x) = \{N_k(\zeta)\}^T \{Q\}_e$

error extremization: $\mathrm{GWS}^N = \int_\Omega \Psi_\beta(x) \mathcal{L}(T^N) \mathrm{d}x \equiv \{0\} \Rightarrow \mathrm{GWS}^h = \mathrm{S}_e \{\mathrm{WS}\}_e$

matrix statement: $\{\mathrm{WS}\}_e = ([\mathrm{DIFF}]_e + [\mathrm{BCs}]_e)\{Q\}_e - \{\mathrm{b(data)}\}_e$

error estimation: $\| e^h \|_E \leq \mathrm{C} \, \ell_e^{2\gamma} \| \mathrm{data} \|_{\mathrm{L}2}^2 , \quad \gamma \equiv \min(\, k+1-m, \, r-m)$

FE *template* pseudo-code

$$\{\mathrm{WS}\}_e = (\mathrm{const})\,(\mathrm{avg})_e \{\mathrm{dist}\}_e^T (\mathrm{metric})_e [\mathrm{FE\ matrix}] \{Q \; or \; \mathrm{data}\}_e$$

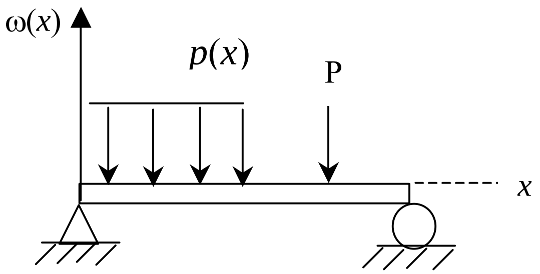
S.6(CM1.10) E-B, T Beams, GWS^h Accuracy/Convergence

GUI creates Matlab script for either theory

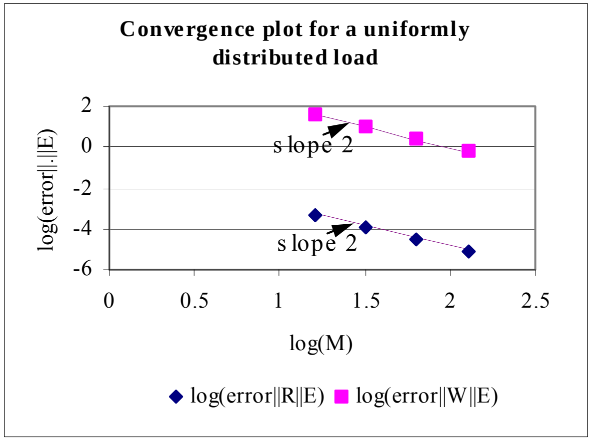
theory:

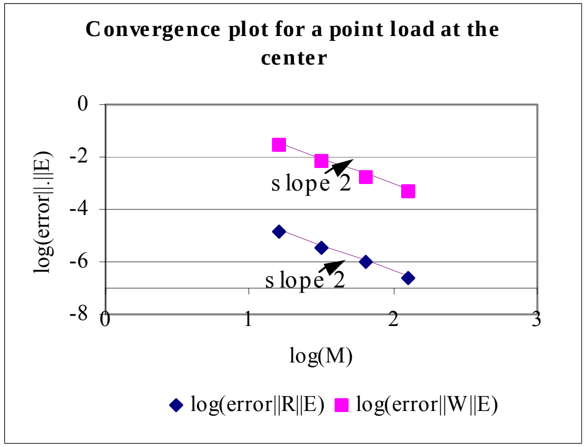
$$\left\| e^h \right\|_E \leq C \ell_e^{2\gamma} \left\| \text{data} \right\|_{L2}^2$$

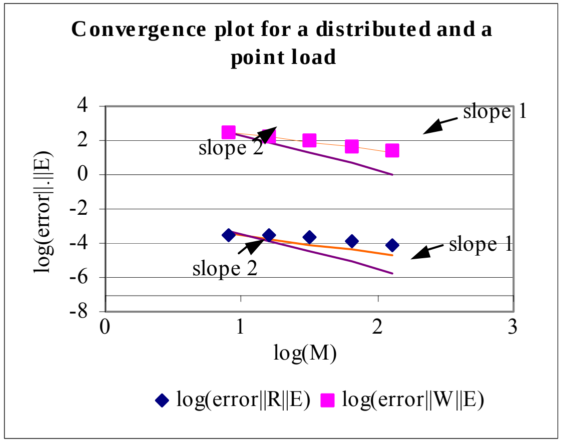
$$\gamma \equiv \min(k, r-1)$$



accuracy/convergence experiments

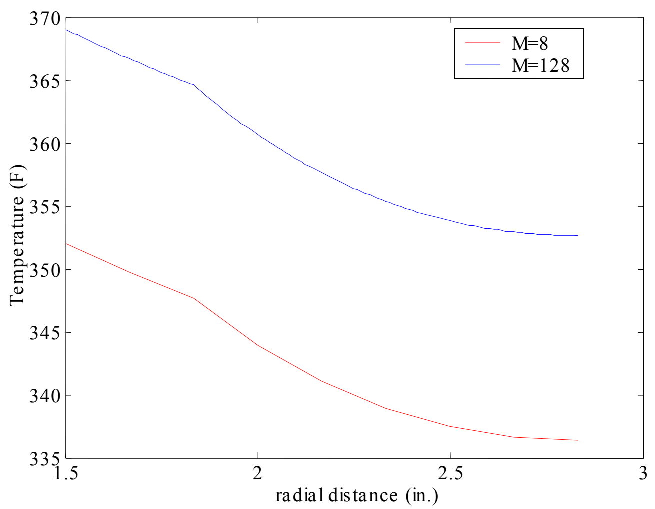


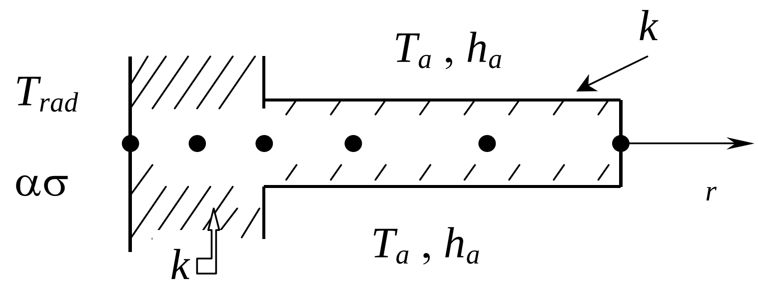




S.7(CM1.27) IC Cylinder, Accuracy, Convergence

GWS^h $\{N_1\}$ basis implementation, temperature distributions





Convergence: $\left\|e^h\right\|_E \leq C l_e^{2\gamma} \left\|\text{data}\right\|_{L2}^2, \;\; \gamma = \min(k, r-1)$

Energy norm convergence data, *k* = 1

Mesh	M	$\|\|T\|\|_E$	slope
h	8	2.36494E+03	
$h/2$	16	2.42266E+03	
$h/4$	32	2.45369E+03	0.89486
$h/8$	64	2.46981E+03	0.94566
$h/16$	128	2.47802E+03	0.97236

Energy norm convergence, [HBC] on all elements

Mesh	M	$\|\|T\|\|_E$	slope
h	8	2.28175700E+03	
$h/2$	16	2.28230949E+03	
$h/4$	32	2.28244772E+03	1.99890
$h/8$	64	2.28248228E+03	1.99972
$h/16$	128	2.28249092E+03	1.99993

S.8(HTn.34) Summary, n-D GWS^h Essence for DE

FE discrete implementation GWS^h for steady DE

"recipe" $\Rightarrow$ analytical transformation of PDE plus BCs to algebraic (computable) form

analogous $\Rightarrow$ transformation methods for linear PDEs

solution^h $\Rightarrow$ parametric function of Re, Gr, Pr, Nu and *data*

error^h $\Rightarrow$ controllable via Ω^h and $\{N_k(\zeta,\eta)\}$ selections

$$\left\| e^h \right\|_E \leq C h^{2\gamma} \left\| \text{data} \right\|_{L2}^2$$

$$\gamma \equiv \min(k+1-m,\, r-m)$$

S.9(CD1.2) Unsteady Scalar Transport

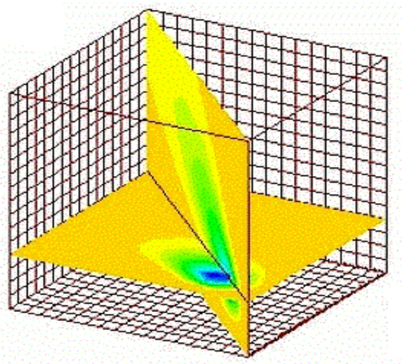
Eulerian non-D description for scalar transport

$$\mathcal{L}(q) = \frac{\partial q}{\partial t} + \mathbf{u}\cdot\nabla q - \mathrm{Pa}^{-1}\nabla\cdot(1+\mathrm{Pa}^t)\nabla q - s = 0\,, \text{ on } \Omega\times t$$

$$\ell(q) = \nabla q\cdot\mathbf{n} + \mathrm{Pb}(q - q_{ref}) + f_n = 0 \quad , \text{ on } \partial\Omega_r\times t$$

$$q(\mathbf{x}_b, t) = q_b(\mathbf{x}_b, t) \quad , \text{ on } \partial\Omega_b\times t$$

$$q(\mathbf{x}, t_0) = q_0(\mathbf{x}) \quad , \text{ on } \Omega\cup\partial\Omega\times t_0$$



Definitions for (x,t), Pa, Pb, Pa^t depend on application

Transport	q	Pa	Pb	Pa^t
heat	Θ	RePr	Nu	$\mathrm{Re}^t/\mathrm{Pr}^t$
mass	Y	ReSc	Pa^{-1}	$\mathrm{Re}^t\mathrm{Sc}^t$
pollutant	Y_α	ReSc	Pa^{-1}	$\mathrm{Re}^t\mathrm{Sc}^t$

$$\mathrm{Re}^t \equiv \left(\frac{\nu^t}{\nu}\right)_{\mathrm{dim}}$$

S.10(CDn.21) Unsteady *n* – D Scalar Transport

Essential ingredients of GWS^h (mDE) + θTS

approximation : $q(\mathbf{x},t) \approx q^N(\mathbf{x},t) \equiv q^h(\mathbf{x},t) = \cup_e q_e(\mathbf{x},t)$

FE basis : $q_e(\mathbf{x},t) = \{N_k(\zeta,\eta)\}^T \{Q(t)\}_e$

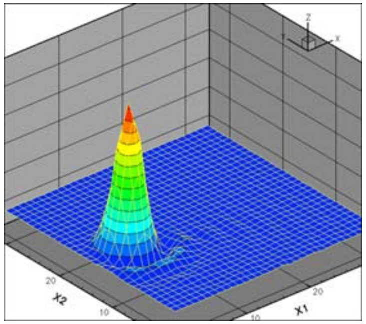
error extremization : $\mathrm{GWS}^N = \int_\Omega \Psi_\beta(\mathbf{x}) \mathcal{L}^m(q^N)\mathrm{d}\tau \equiv \{0\} \Rightarrow \mathrm{GWS}^h$

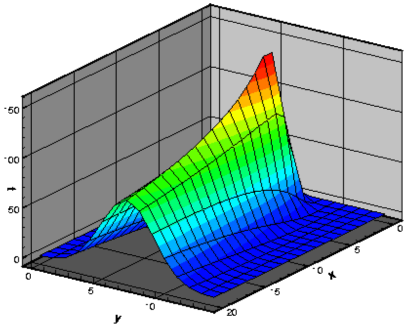
matrix statement : $\mathrm{GWS}^h + \theta\mathrm{TS} \Rightarrow [m\mathrm{JAC}]\{\Delta Q\} = -\Delta t\{m\mathrm{RES}\}$

$[m\mathrm{JAC}] = \mathrm{S}_e([m\mathrm{JAC}]_e), \quad \{m\mathrm{RES}\} = \mathrm{S}_e(\{m\mathrm{RES}\}_e)$

asymptotic convergence : $\|e^h(t)\|_E \le Ch_e^{f(k,\mathrm{Pe},\beta)}\|\mathrm{data}\|_{\mathrm{L}2}^2 + C_t\Delta t^{f(\theta)}\|q_0\|_E$

error spectra : $U_\omega \ \& \ G \Rightarrow f(\omega,k,h,\Delta t,\theta,\alpha,\beta,\gamma)$



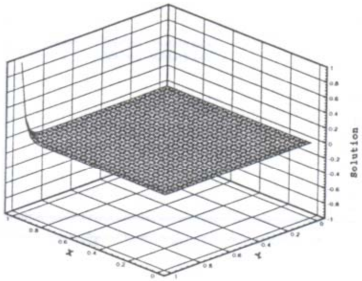


Template pseudo-code converts theory ⇒ practice

$\mathrm{GWS}^h(m\mathrm{D}E) + \theta\mathrm{TS} \Rightarrow \mathrm{S}_e\{\mathrm{WS}\}_e \equiv \{0\}$

$\{\mathrm{WS}\}_e \equiv (\mathrm{const})(\mathrm{avg})_e\{\mathrm{dist}\}_e(\mathrm{metric})[\mathrm{Matrix}]_e\{Q \ or \ \mathrm{data}\}_e$

$[\mathrm{JAC}]_e \equiv \partial\{\mathrm{WS}\}_e / \partial\{Q\}_e$

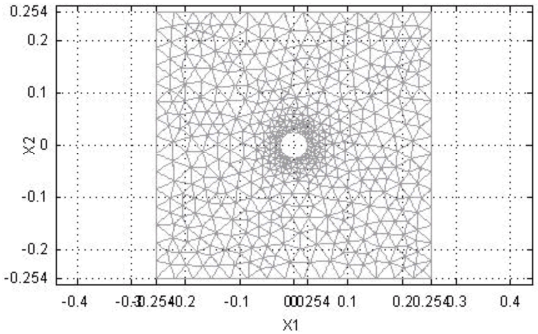


S.11(CM*n*.19) Plane Stress: Plate with a Hole

GWS^h for DP and/or DE extremum, plane stress, n = 2

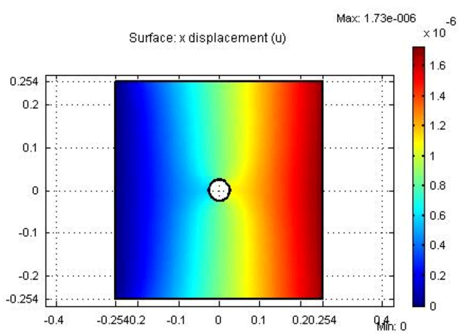
$$[\text{Matrix }(\mathbf{E},\upsilon)]\begin{Bmatrix} U \\ V \end{Bmatrix} = \{\mathbf{R}\ (\varepsilon_0, \tau_0, \mathbf{T}, \mathbf{P}, \rho\mathbf{g})\}$$

meshing, Ω^h

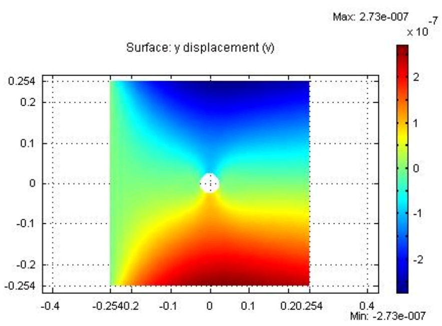


Computer lab design study

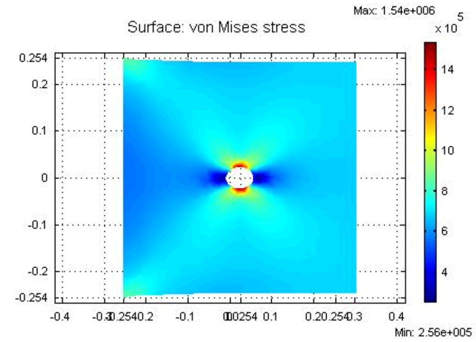
geometry: plate with hole in tension
data: L, D, **T**, BCs
solution: $u^h\ (x, y)$, $v^h\ (x, y)$
interpretation: von Mises stress concentration



x displacement, u^h



y displacement, v^h



von Mises stress

S.12(CMn.23) Streamfuction-Vorticity Navier-Stokes

For n = 2: $\mathbf{u} = \nabla \times \psi\hat{\mathbf{k}}$ *and* $\omega \equiv \nabla \times \mathbf{u} \cdot \hat{\mathbf{k}}$

DM: $\nabla \cdot \mathbf{u} = \nabla \cdot \nabla \times \psi\hat{\mathbf{k}} = 0$ identically

$\hat{\mathbf{k}} \cdot \nabla \times \mathrm{DP}$: $\omega_t + \nabla \times \psi\hat{\mathbf{k}} \cdot \nabla\omega - \mathrm{Re}^{-1}\nabla^2\omega = 0$

kinematics: $\omega = \nabla \times \mathbf{u} \cdot \hat{\mathbf{k}} = \nabla \times \nabla \times \psi\hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = -\nabla^2\psi$

Steady-state N-S PDEs + BCs:

$$\mathcal{L}(\omega) = -\mathrm{Re}^{-1}\nabla^2\omega + \nabla \times \psi\hat{\mathbf{k}} \cdot \nabla\omega = 0$$

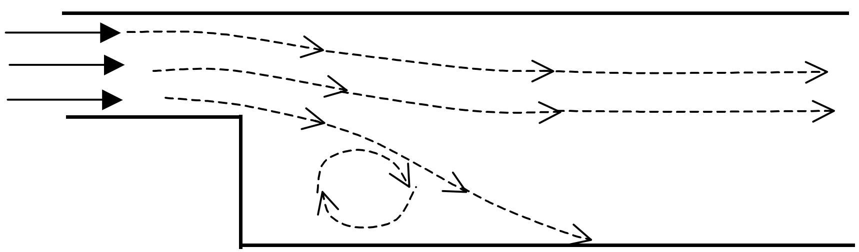
$$\mathcal{L}(\psi) = -\nabla^2\psi - \omega = 0$$

$\partial\Omega_{\mathrm{in}}$: $\mathbf{u}(y, x_{\mathrm{in}}) \Rightarrow \omega_{\mathrm{in}}, \psi_{\mathrm{in}}$ via definitions

$\partial\Omega_{\mathrm{out}}$: $\hat{\mathbf{n}} \cdot \nabla(\omega, \psi) = 0$

$\partial\Omega_{\mathrm{wall}}$: $\psi = \psi_w =$ constant

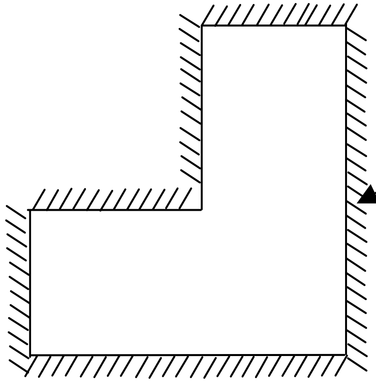
$\hat{\mathbf{n}} \cdot \nabla\omega = f_w(\psi, \omega)$



S.13(CMn.31) Mechanical Vibrations Normal Modes

Transverse vibrations of a plate

$$\mathrm{d}\mathbf{P}: \quad \frac{\partial^2 y}{\partial t^2} - \nabla \cdot f(E, \upsilon) \nabla y = 0$$

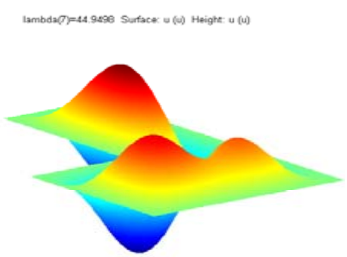


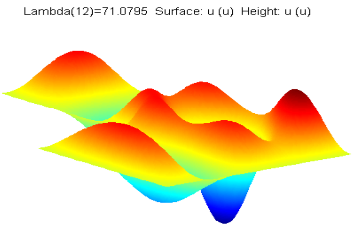
normal mode solution: $y(\mathbf{x}, t) = \mathrm{Q}(\mathbf{x})\, e^{i\omega t}$

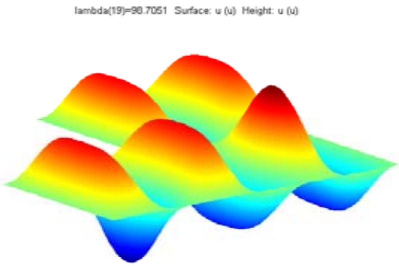
GWS^h for eigenmodes: $\left[[\mathrm{STIFF}] - \omega^2 [\mathrm{MASS}] \right] \{Q\} = \{0\}$

homogeneous BCs: $\det([\mathrm{MASS}]^{-1} [\mathrm{STIFF}] - \omega_i^2 [\mathrm{I}]) = \{0\}$

GWS^h normal mode solutions, $\omega_i^h = 45, 71, 99$ for $i = 7, 12, 19$







S.14(FE.1) Engineering Simulation

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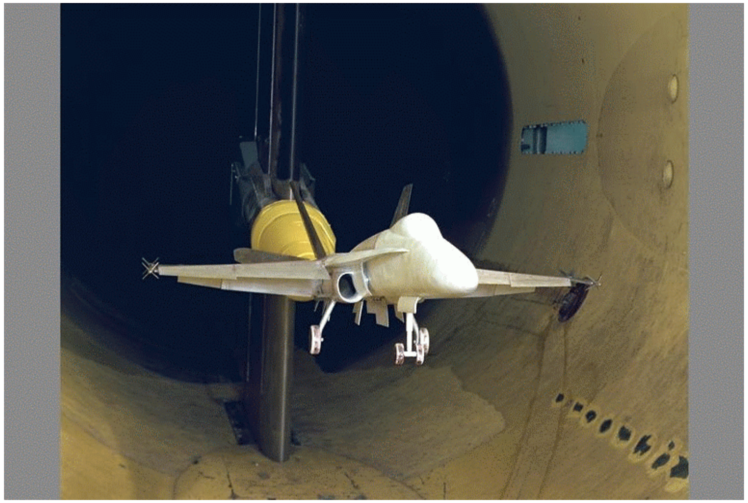

Computational Laboratory:

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  physics model error
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