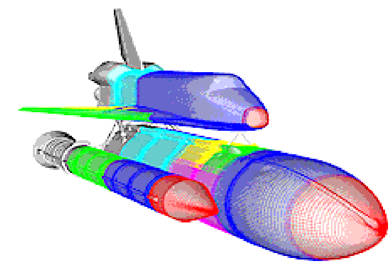
FM.1 Fluid Mechanics, Simplified Analyses

Conservation principles, *Eulerian* viewpoint

$$\mathrm{D}M \;=\; \frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho\, d\tau \;+\; \oint_{\mathrm{CS}} \rho \mathbf{V}\cdot\hat{\mathbf{n}}\, d\sigma \;=\; 0$$

$$\mathrm{D}\mathbf{P} \;\Rightarrow\; \frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho \mathbf{V}\, d\tau + \oint_{\mathrm{CS}} \mathbf{V}\rho\mathbf{V}\cdot\hat{\mathbf{n}}\, d\sigma = \int_{\mathrm{CV}} \rho \mathbf{B}\, d\tau + \int_{\mathrm{CS}} \mathbf{T}\, d\sigma$$

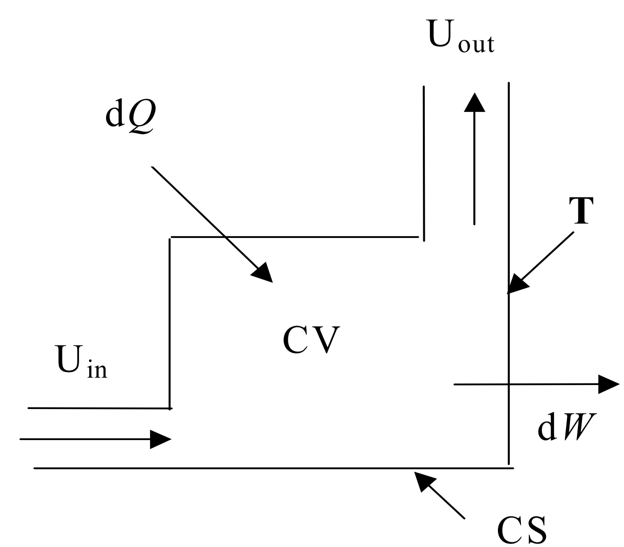


$$\mathrm{D}E \;\Rightarrow\; \frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho e\, d\tau + \oint_{\mathrm{CS}} \left(e + p/\rho\right)\rho\mathbf{V}\cdot\hat{\mathbf{n}}\, d\sigma = \int_{\mathrm{CV}} s\, d\tau + \oint_{\mathrm{CS}} \left(W - \mathbf{q}\cdot\hat{\mathbf{n}}\right) d\sigma$$

Control volume, uni-directional flow

$\mathrm{D}M$, $\mathrm{D}E$ $\Rightarrow$ algebraic equations
physics $\Rightarrow$ heat added, work done
$\mathrm{D}\mathbf{P}$ $\Rightarrow$ reaction force $\mathbf{T}$

data: velocity in, heat added, fluid properties
output: velocity out, work done, reaction force



FM.2 Fluid Mechanics, Constitutive Closure Models

Conservation principles, arbitrary Eulerian CV

$$\mathrm{D}M : \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0$$

$$\mathrm{D}\mathbf{P} : \frac{\partial \rho \mathbf{V}}{\partial t} + \nabla \cdot \rho \mathbf{V}\mathbf{V} = \rho \mathbf{g} + \nabla \mathbf{T}$$

$$\mathrm{D}E : \frac{\partial \rho e}{\partial t} + \nabla \cdot (\rho e + p)\mathbf{V} = s - \nabla \cdot \mathbf{q}$$

Constitutive closure models ⇒ Navier-Stokes equations

Stokes viscosity model

$$\mathbf{T} \equiv -p\mathbf{I} + \mu \cdot \nabla \mathbf{V} - \frac{2\lambda}{3}(\nabla \cdot \mathbf{V})\delta$$

Fourier conduction model

$$\mathbf{q} = -k\nabla T$$

perfect gas, internal energy

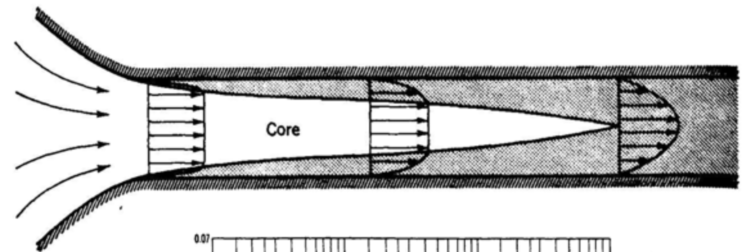
$$p = \rho \mathrm{R} T, \;\; e = c_v T, \; c = \sqrt{\gamma \mathrm{R} T}$$

where: μ, λ, k, R, c_v are fluid properties (data)

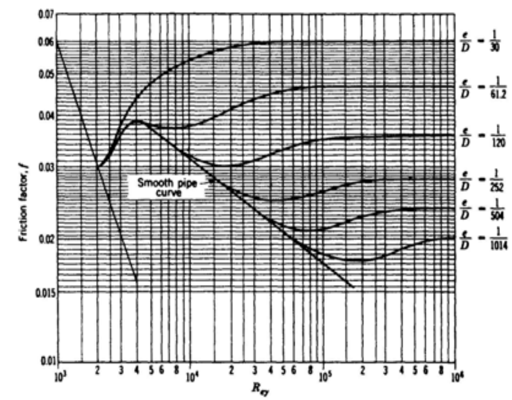
FM.3 Fluid Mechanics, Navier-Stokes Equations

One-dimensional simplifications

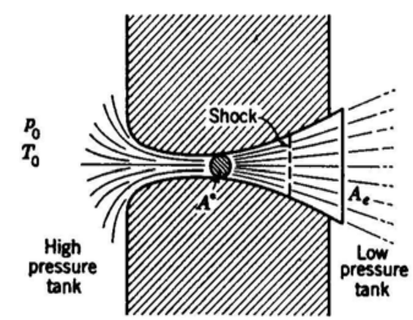
textbook pipe flow
physics – laminar, D*E* = 0
D**P** ⇒ ordinary differential equations



practical pipe flow
physics - turbulent flow
D**P** ⇒ data correlations (Re, ε/D)
concepts – similitude, roughness
D*M*, D*E* ⇒ piping networks, pumping



compressible flow
physics – isentropic (μ = 0 = *k*)
– perfect gas (γ, R)
D*M*, D*E* ⇒ exponential algebraic equations
concepts – sonic throat, shock
D**P** ⇒ reaction force



FM.4 Navier-Stokes Equations, non-Dimensional

non-D, incompressible laminar flow + heat transfer

$$\mathrm{D}M: \quad \nabla \cdot \mathbf{u} = 0$$

$$\mathrm{D}\mathbf{P}: \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla P - \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u} - \frac{\mathrm{Gr}}{\mathrm{Re}^2} \Theta \hat{\mathbf{g}} = 0$$

$$\mathrm{D}E: \quad \frac{\partial \Theta}{\partial t} + (\mathbf{u} \cdot \nabla)\Theta - \frac{1}{\mathrm{Re\,Pr}} \nabla^2 \Theta - s = 0$$

on $\Omega \subset \Re^n$ + BCs + IC

closure models, non-D groups

closure models:

$$\rho \mathbf{g} \Rightarrow \mathrm{Gr\,Re}^{-2} \Theta \hat{\mathbf{g}}$$

$$\mathbf{T} \Rightarrow \mathrm{Re}^{-1}$$

$$\mathbf{q} \Rightarrow (\mathrm{Re\,Pr})^{-1}$$

$$\mathrm{BCs} \Rightarrow \mathrm{Nu}$$

non-D variables:

$$\Theta = (T - \mathrm{T}_{min}) / \Delta \mathrm{T}$$

$$\mathbf{u} = \mathbf{V} / \mathrm{U}_\infty$$

$$P = p / \rho_0 \mathrm{U}_\infty^2$$

Grashoff $= \mathrm{Gr} = \rho_0^2 \beta g \Delta \mathrm{T} \mathrm{L}^3 / \mu^2$

Reynolds $= \mathrm{Re} = \rho_0 \mathrm{UL} / \mu$

Prandtl $= \mathrm{Pr} = \mathrm{c}_p \mu / k$

Nusselt $= \mathrm{Nu} = h\mathrm{L} / k$

Rayleigh $= \mathrm{Ra} = \dfrac{\beta g k}{\rho \mathrm{c}_p \mu \mathrm{U}^2} (\Delta \mathrm{T})$

FM.5 Navier-Stokes, Aerodynamics Simplification

Characteristics of aerodynamic flows

aerodynamic shapes
- flowfield is uni-directional
- farfield is undistributed

large Reynolds number, $Re/L > 10^6$
- viscous effects strictly local

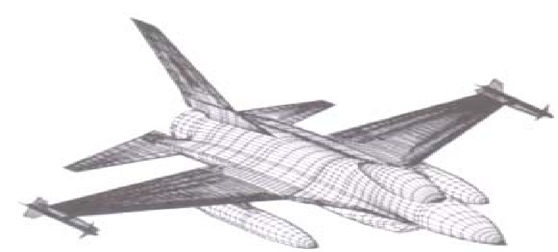
Conservation principle simplifications

farfield described by potential flow $\Rightarrow \mathbf{u} = -\nabla\phi$

$$\mathrm{D}M\!: \nabla\cdot\rho\mathbf{u} \Rightarrow -\nabla\cdot\rho\nabla\phi = 0$$

$$\int \mathrm{D}\mathbf{P}\cdot d\mathbf{\Psi} \Rightarrow p(\mathbf{x}) = p_\infty - \frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}$$

Bernoulli pressure



nearfield DM + D**P** can be Re-ordered
$\Rightarrow$ boundary layer form of N-S

FM.6 Aerodynamics – Potential Flow

Inviscid, irrotational steady flow

subsonic – transonic – supersonic: Mach $\equiv M = U/a$, $a = \sqrt{\gamma R T} = \sqrt{U_\infty / (\gamma R T)}$

$$M_a < 0.3: \quad DM = \nabla \cdot \mathbf{u} \Rightarrow -\nabla^2 \phi = 0$$

$$M_a \approx 1: \quad DM = \nabla \cdot \rho \mathbf{u} \Rightarrow \left(1 - M_\infty^2\right)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - M_\infty^2 \left[\frac{1+\gamma}{U_\infty}\right]\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x^2} = 0$$

$$M_a > 1: \quad DM = \nabla \cdot \rho \mathbf{u} \Rightarrow \left(1 - M_x^2\right)\frac{\partial^2 \phi}{\partial x^2} + \left(1 + M_y^2\right)\frac{\partial^2 \phi}{\partial y^2} - \frac{2}{a^2}\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial x \partial y} = 0$$

Mach ≤ 0.3

Mach ≈ 1

Mach > 1

FM.7 Subsonic Potential Flow

n-D subsonic aerodynamics

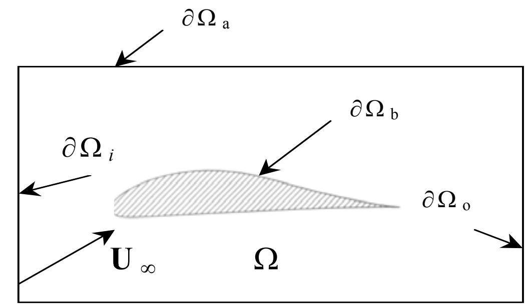
DM:

$$\mathrm{L}(\phi) = -\nabla^2\phi = 0 \qquad , on\ \Omega \subset \Re^n$$

$$\ell(\phi) = \mathbf{u}\cdot\hat{\mathbf{n}} = -\hat{\mathbf{n}}\cdot\nabla\phi = 0 \qquad , on\ \partial\Omega_{\text{body}}$$

$$= \mathbf{U}_\infty \cdot \hat{\mathbf{n}} \qquad , on\ \partial\Omega_{\text{inflow}}$$

$$\phi(\mathbf{x}_b) = 0 \qquad , on\ \partial\Omega_{\text{outflow}}$$

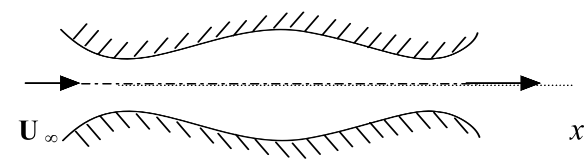


$$GWS^h = \mathrm{S}_e\{WS\}_e = 0, \quad \{WS\}_e = (\ \ )(\ \ )\{\ \ \}(-1)[M2KK]\{PHI\} + BCs$$

quasi-one dimensional ducted flow

DM:

$$\mathrm{L}(\phi) = -A(x)\frac{d^2\phi}{dx^2} - \frac{dA}{dx}\frac{d\phi}{dx} = 0$$



$$\begin{aligned}\{WS(\phi^h)\}_e = &(\ \ )(\ \ )\{AREA\}(-1)[A3011]\{PHI\}\\ &+(\ \ )(\ \ )\{AREA\}(-1)[A3101]\{PHI\}\\ &+(-)(\ \ )\{AREA\}(-1)[A3101]\{PHI\}\\ &+(UDOTN)(\ \ )\{AREA\}(\ \ )[ONE]\{\ \ \}\end{aligned}$$

DE:

$$\mathrm{L}(p) = p - p_\infty + \frac{1}{2}\rho(d\phi/dx)^2$$

$$\begin{aligned}\{WS(P)\}_e = &(\ \ )(\ \ )\{\ \ \}(1)[A200]\{P\} + (-)(\ \ )\{\ \ \}(1)[A200]\{P_{inf}\}\\ &+(1/2, RHO)(\ \ )\{PHI\}(-1)[A3101]\{PHI\}\end{aligned}$$

FM.8 Aerodynamics, Weak Interaction Theory

Farfield, subsonic-transonic potential flow assumption

D*M*:

$$\mathrm{L}(\phi) = (1 - \mathrm{M}_\infty^2)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

$$\ell(\iota) = \hat{\mathbf{n}} \cdot \nabla\phi - \mathrm{U}_\infty \hat{\mathbf{i}} \cdot \hat{\mathbf{n}} = 0$$

D*E*:

$$p(\mathbf{x}_\delta) = p_\infty - \rho\nabla\phi \cdot \nabla\phi / 2$$

Nearfield, boundary layers wash aerosurfaces

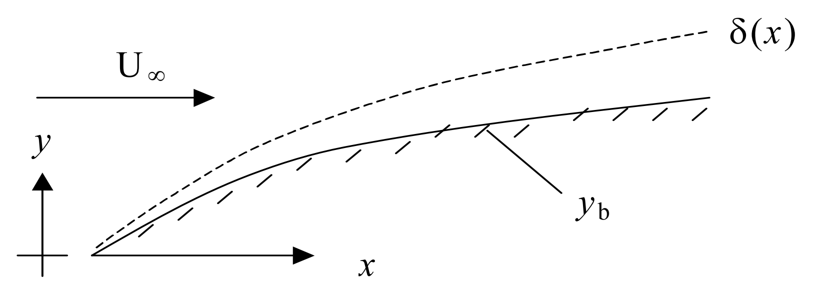
viscous, turbulent effects dominate

D*M*:

$$\nabla \cdot \mathbf{u} = 0$$

D**P**:

$$\mathbf{u} \cdot \nabla\mathbf{u} = \frac{-1}{\rho_0}\nabla p + \nabla\mathbf{T}$$



in the region $y_b(x) \le y(x) \le \delta(x)$

$\delta(x) \equiv$ boundary layer thickness

$\mathbf{T}$ = viscous + turbulent effects

FM.9 Aerodynamics, Boundary Layer Flow

Reynolds ordering of Navier-Stokes, subsonic, $n = 2$

known scales: U_∞, L, $\delta(x)$

non-D ordering: $u/U_\infty \approx O(1)$, $x/L \approx O(1)$, $\delta/L << O(1)$

DM:

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow O(1/1) + O(v/\delta) = 0, \quad \text{hence}, \; v/U_\infty \approx O(\delta)$$

D**P**:

$$\mathbf{u} \cdot \nabla \mathbf{u} + \mathrm{Eu} \nabla p - \mathrm{Re}^{-1} \nabla^2 \mathbf{u} = 0$$

$$\hat{\mathbf{i}} \Rightarrow O(1 \cdot 1/1) + O(\delta \cdot 1/\delta) + \mathrm{Eu}\; O(p/1) - \mathrm{Re}^{-1} O(1/1 \cdot 1 + 1/\delta \cdot \delta) = 0$$

$$\text{keeping } \; O(1) \Rightarrow \mathrm{Eu}\; \partial p/\partial x \Rightarrow O(1), \;\; \mathrm{Re}^{-1} \Rightarrow O(\delta^2), \; \frac{\partial^2 u}{\partial x^2} \Rightarrow O(\delta^2)$$

$$\hat{\mathbf{j}} \Rightarrow O(1 \cdot \delta/1) + O(\delta \cdot \delta/\delta) + \mathrm{Eu}\; O(1/\delta) - \mathrm{Re}^{-1} O(\delta/1 \cdot 1 + \delta/\delta \cdot \delta) = 0$$

$$\text{everything } \; O(\delta) \Rightarrow \text{hence } \; \mathrm{Eu}\; \partial p/\partial y = O(\delta), \text{ then } \; p(x,y) \Rightarrow p(x)$$

recall:

$$\mathrm{Re} = \rho_\infty U_\infty L / \mu_\infty$$

$$\mathrm{Eu} = p_\infty / \rho_\infty U_\infty^2$$

FM.10 Boundary Layer Form of Navier-Stokes

Summary, Reynolds ordering of N-S, $n = 2$, steady subsonic BL

DP$_y$: pressure through BL is constant
$\Rightarrow P(x)$ from potential farfield DM solution

DP$_x$: $\partial^2 u/\partial x^2$ is $O(\delta^2)$, hence negligible, Re $= O(\delta^{-2}) >> 1$
$\Rightarrow$ BL DP$_x$ is initial-value on $x \geq x_0$

DM: $\partial v/\partial y = - \partial u/\partial x$, hence initial value on $0 \leq y < \delta(x)$

Conservation principles for laminar BL flow simplify to

$$\mathrm{L}(u) = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \mathrm{Eu}\,\frac{\mathrm{d}P_I}{\mathrm{d}x} - \frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial y^2} = 0$$

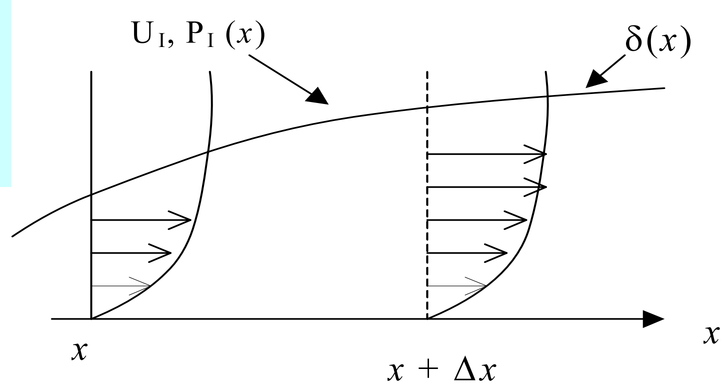
$$\mathrm{L}(v) = \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} = 0$$

BCs:

$$u(x, y = 0) = 0 = v(x, y = 0)$$

$$\left.\frac{\partial u}{\partial y}\right|_{x, y/\delta > 1} = 0$$

$$\delta(x) \equiv y\big(u(x)/\mathrm{U}_I(x)\big) \equiv 0.99$$



FM.11 Boundary Layer Flow, Turbulence

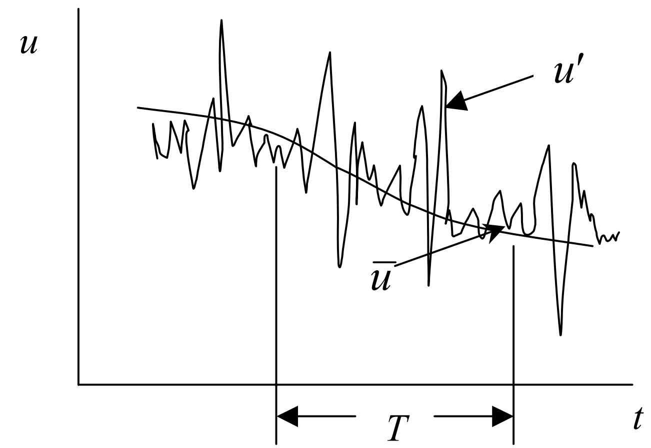
BL form of NS valid only for Re >> 1

aircraft	Mach	U_∞ (m/s)	L (m)	Re	Re/L
commuter	0.3	125	10	3E07	O(E06)
wide body	0.9	250	40	2E08	O(E06)

BL flows will be turbulent (!)

resolution of BL velocity components

$$u(\mathbf{x},t) \equiv \overline{u}(\mathbf{x}) + u'(\mathbf{x},t)$$



time-averaging

$$\overline{u}(\mathbf{x}) \equiv \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} u(\mathbf{x},\tau)\,d\tau$$

$$\overline{u'} = 0$$

FM.12 Boundary Layer Flow, Reynolds Stress

Time averaging of BL DM and DP

DM: both terms linear, hence $\nabla \cdot \overline{\mathbf{u}} = 0 = \nabla \cdot \mathbf{u}'$

DP$_x$: non-linear convection term generates a new contribution

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \Rightarrow \frac{\partial}{\partial x}(uu) + \frac{\partial}{\partial y}(vu) \quad \text{via D}M$$

$$\overline{uu} \Rightarrow \overline{u}\;\overline{u} + \overline{u'u'}$$
$$\overline{vu} \Rightarrow \overline{v}\;\overline{u} + \overline{v'u'}$$

Reynolds ordering confirms that $O\left(\overline{u'u'}\right) \approx O\left(\overline{v'u'}\right) \approx O(\delta)$

$$\frac{\partial}{\partial x}\left(\overline{u}\;\overline{u} + \overline{u'u'}\right) \Rightarrow O\left(1 \cdot 1/1 + \delta/1\right)$$
$$\frac{\partial}{\partial y}\left(\overline{v}\;\overline{u} + \overline{v'u'}\right) \Rightarrow O\left(\delta \cdot 1/\delta + \delta/\delta\right)$$

hence:

Reynolds normal stress $\overline{u'u'}$ contribution negligible

Reynolds shear stress $\overline{v'u'}$ contribution must be included

FM.13 Boundary Layer Flow, Turbulence Modeling

Reynolds kinematic shear stress modeled after Stokes, FM.2

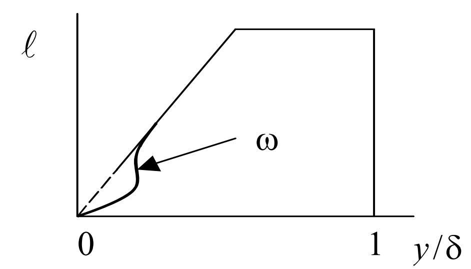
$$\overline{v'u'} \equiv -\nu^t \frac{\partial \overline{u}}{\partial y} \quad , \quad \nu^t \equiv \text{turbulent "eddy" viscosity, units are } \mathrm{L}^2/t$$

Prandtl mixing length model

$$\nu^t \equiv (\omega \ell)^2 \left| \frac{\partial \overline{u}}{\partial y} \right| \Rightarrow (L^2)(1/t)$$

where: $\ell \equiv$ mixing length

ω = van Driest near wall damping



Turbulent kinetic energy-dissipation model

$$\nu^t \equiv C_\mu k^2/\varepsilon \Rightarrow (\mathrm{L}/t)^4 (t^3/\mathrm{L}^2)$$

where:

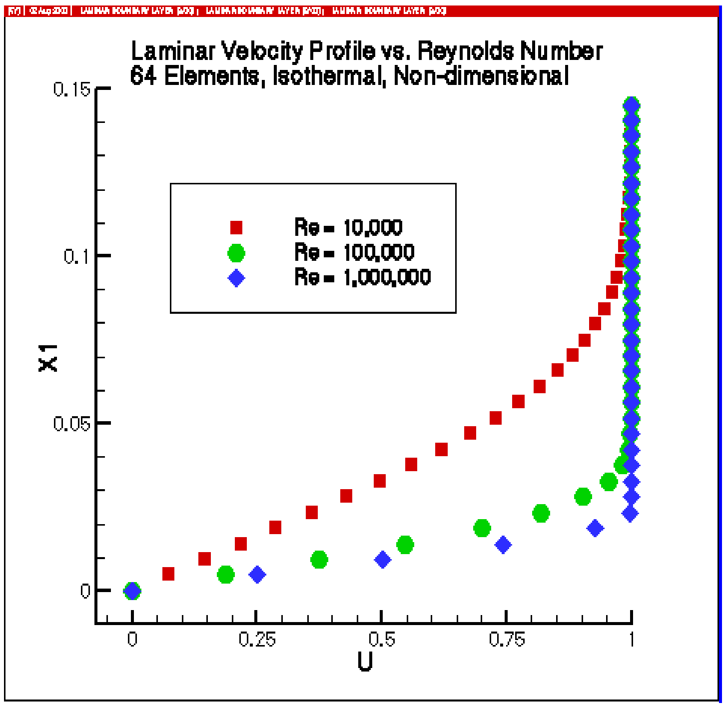
$$k \equiv \frac{1}{2}\left(\overline{\mathbf{u}'\cdot\mathbf{u}'}\right) = \frac{1}{2}\left(\overline{u'u'} + \overline{v'v'} + \overline{w'w'}\right)$$

$$\varepsilon \equiv \frac{2\nu}{3}\overline{\left(\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_i}{\partial x_j}\right)}\delta_{jk}$$

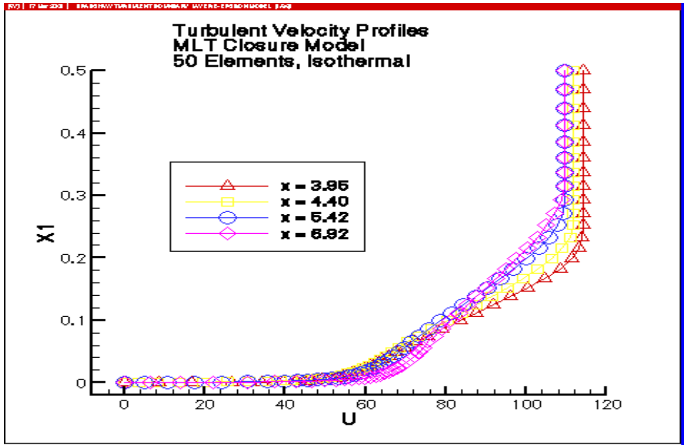
and: L (k) and L (ε) BL forms augment BL DM & DP_x

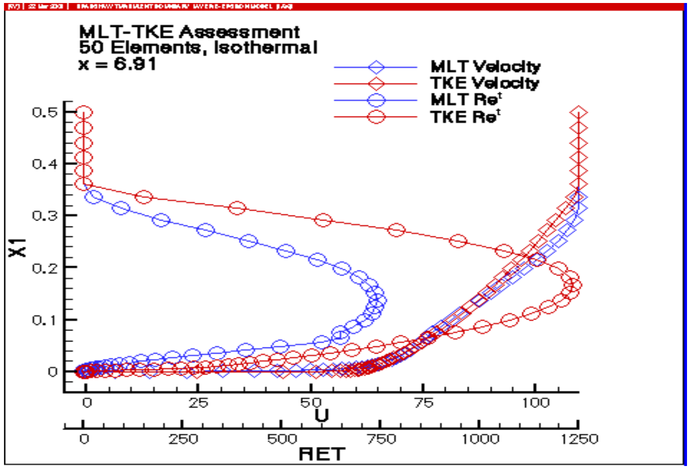
FM.14 Aerodynamic Boundary Layers, Laminar & Turbulent

Downstream velocity profiles, laminar, $10^4 \leq Re/L \leq 10^6$



Turbulent BL, $Re/L \approx 10^5$





FM.15 Navier-Stokes, Computational Fluid Dynamics

Fluid-thermal-structural system design uses "CFD"

for Reynolds-averaged, turbulent, unsteady incompressible flow

DM: $\nabla \cdot \mathbf{u} = 0$

D$\mathbf{P}$: $$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \mathrm{Eu}\, \nabla p - \nabla \cdot \left(\mathrm{Re}^{-1} + \nu^t\right) \nabla \mathbf{u} + \frac{\mathrm{Gr}}{\mathrm{Re}^2} \Theta \hat{\mathbf{g}} = \mathbf{0}$$

DE: $$\frac{\partial \Theta}{\partial t} + \mathbf{u} \cdot \nabla \Theta - \mathrm{Re}^{-1} \nabla \cdot \left(\mathrm{Pr}^{-1} + \nu^t / \mathrm{Pr}^t\right) \nabla \Theta + s_\Theta = \mathbf{0}$$

DE': $$\frac{\partial k}{\partial t} + \mathbf{u} \cdot \nabla k - \mathrm{Re}^{-1} \nabla \cdot \left(\mathrm{Pr}^{-1} + \nu^t / \mathrm{Pr}^t\right) \nabla k + \mathbf{T} \nabla \mathbf{u} - \varepsilon = \mathbf{0}$$

DE_{m}': $$\frac{\partial \varepsilon}{\partial t} + \mathbf{u} \cdot \nabla \varepsilon - \mathrm{Re}^{-1} \nabla \cdot \left(\mathrm{C}_\varepsilon \nu^t / \mathrm{Pr}^t\right) \nabla \varepsilon + \mathrm{C}_\varepsilon^1 \mathbf{T}_\mathbf{k}^\varepsilon \frac{\varepsilon}{k} \nabla \mathbf{u} - \mathrm{C}_\varepsilon^2 \varepsilon^2 / k = \mathbf{0}$$

where:

non-D groups were defined on FM.4
turbulence closure, generalization of BL development, FM.12, $\nu^t \equiv \mathrm{C}_\mu k^2 / \varepsilon$

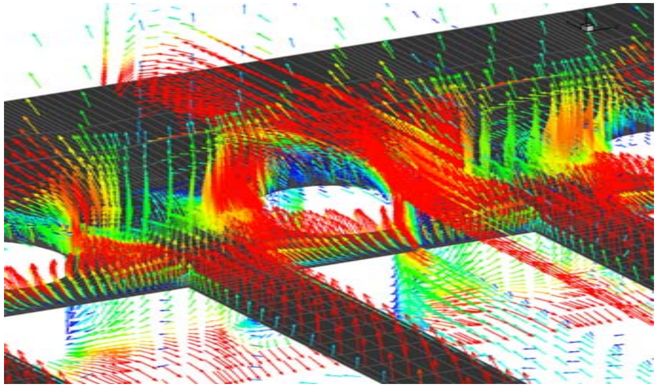
algebraic Reynold stress model

$$-\mathbf{T} = -\frac{2}{3} k \boldsymbol{\delta} + \nu^t \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right) + \ldots$$

FM.16 Navier-Stokes, CFD, Algorithm Issues

NS turbulent flow conservation law PDEs

initial-value
explicitly non-linear!
multiple DOF/node!
as given, are ill-posed!
intrinsically unstable for Re >> 1



Commercial CFD codes contain required capabilities

have 15-20 year development history
are based on FD, FV and/or FE discrete methods
learning curve is *steep*!
chances for error are numerous
output interpretation requires color graphics & animations

FM.17 Incompressible N-S, Well-Posedness

Consider unsteady isothermal laminar NS

DM: $\nabla \cdot \mathbf{u} = 0$

D**P**: $\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \mathrm{Eu} \nabla p - \mathrm{Re}^{-1} \nabla^2 \mathbf{u} = 0$

$\Rightarrow$ 4 PDEs on $\mathbf{u}$, none on p !

Mathematically, DM **is a *constraint* on solutions to** DP

theories to enforce the constraint include

pseudo-compressibility: D$M \Rightarrow \beta^{-1} p_t + \nabla \cdot \mathbf{u} = 0$

pressure projection: $\left\| \nabla \cdot \mathbf{u}^h \right\| \Rightarrow \varepsilon > 0$, iteratively

vector field theory: DM guarantees $\mathbf{u} = \nabla \times \mathbf{\Psi}$

$\nabla \times$D**P** eliminates pressure appearance

$\Rightarrow$ for $n = 2$, produces streamfunction-vorticity formulation

FM.18 Streamfuction-Vorticity Navier-Stokes

For $n = 2$: $\mathbf{u} = \nabla \times \psi\hat{\mathbf{k}}$ *and* $\omega \equiv \nabla \times \mathbf{u} \cdot \hat{\mathbf{k}}$

DM: $\nabla \cdot \mathbf{u} = \nabla \cdot \nabla \times \psi\hat{\mathbf{k}} = 0$ identically

$\hat{\mathbf{k}} \cdot \nabla \times \mathrm{DP}$: $\omega_t + \nabla \times \psi\hat{\mathbf{k}} \cdot \nabla\omega - \mathrm{Re}^{-1}\nabla^2\omega = 0$

kinematics: $\omega = \nabla \times \mathbf{u} \cdot \hat{\mathbf{k}} = \nabla \times \nabla \times \psi\hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = -\nabla^2\psi$

Steady-state N-S PDEs + BCs:

$$\mathcal{L}(\omega) = -\mathrm{Re}^{-1}\nabla^2\omega + \nabla \times \psi\hat{\mathbf{k}} \cdot \nabla\omega = 0$$

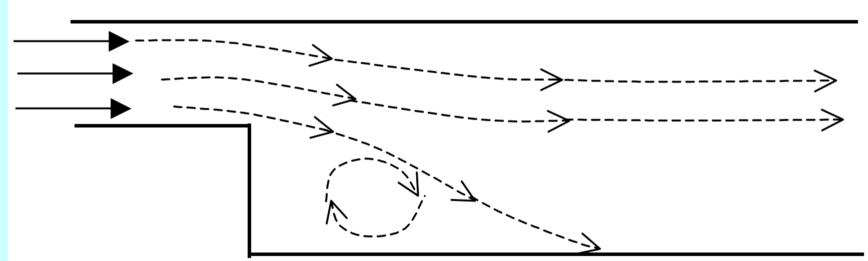
$$\mathcal{L}(\psi) = -\nabla^2\psi - \omega = 0$$

$\partial\Omega_{\mathrm{in}}$: $\mathbf{u}(y, x_{\mathrm{in}}) \Rightarrow \omega_{\mathrm{in}}, \psi_{\mathrm{in}}$ via definitions

$\partial\Omega_{\mathrm{out}}$: $\hat{\mathbf{n}} \cdot \nabla(\omega, \psi) = 0$

$\partial\Omega_{\mathrm{wall}}$: $\psi = \psi_w =$ constant

$\hat{\mathbf{n}} \cdot \nabla\omega = f_w(\psi, \omega)$



FM.19 GWS^h, Streamfunction-Vorticity NS, n = 2

Galerkin weak statements:

for $q^N(x,y) \equiv \sum_{\alpha} \Psi_\alpha(\mathbf{x}) Q_\alpha, \quad q = \{\omega, \psi\}^T$

$$\mathrm{GWS}^N(\omega) \equiv \int_\Omega \Psi_\beta(\mathbf{x}) \mathrm{L}(\omega^N) d\tau = \int_\Omega \Psi_\beta \left[-\mathrm{Re}^{-1} \nabla^2 \omega^N + \nabla \times \psi^N \hat{\mathbf{k}} \cdot \nabla \omega^N \right] d\tau$$

$$= \int_\Omega \left[\mathrm{Re}^{-1} \nabla \Psi_\beta \cdot \nabla \Psi_\alpha \mathrm{OMG}_\alpha + \Psi_\beta \nabla \times \Psi_\gamma \mathrm{PSI}_\gamma \cdot \nabla \Psi_\alpha \mathrm{OMG}_\alpha \right] d\tau + \mathrm{BC}$$

$$\mathrm{GWS}^N(\psi) = \int_\Omega \Psi_\beta L(\psi^N) d\tau = \int_\Omega [\nabla \Psi_\beta \cdot \nabla \Psi_\alpha \mathrm{PSI}_\alpha - \Psi_\beta \Psi_\alpha \mathrm{OMG}_\alpha]\, d\tau + \mathrm{BC}$$

thus:

$$\mathrm{GWS}^N \Rightarrow \mathrm{GWS}^h = \mathrm{S}_e \{\mathrm{WS}\}_e \equiv 0$$

$$\{\mathrm{WS}(\omega^h)\}_e = \mathrm{Re}^{-1} [\mathrm{B2}KK]_e \{\mathrm{OMG}\}_e + \{\mathrm{PSI}\}_e^T [\mathrm{B3}K0K]_e \{\mathrm{OMG}\}_e$$

$$+ \mathrm{Re}^{-1} [\mathrm{A200}]_e \{ f_w(\psi^h, \omega^h \}_e$$

$$\{\mathrm{WS}(\psi^h)\}_e = [\mathrm{B2}KK] \{\mathrm{PSI}\}_e - [\mathrm{B200}]_e \{\mathrm{OMG}\}_e + [\mathrm{A200}]_e \{\mathrm{U}_w\}_e$$

FM.20 GWS^h Details, Streamfunction-Vorticity NS

GWS ^{h} for ω^h involves $\nabla \times \psi \hat{\mathbf{k}} \cdot \nabla \omega$

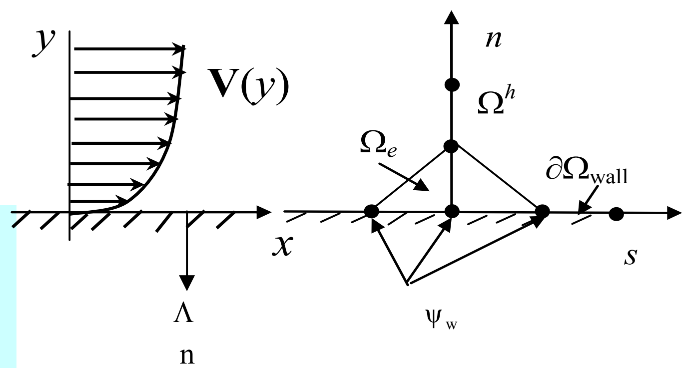
$$\nabla \times \psi \hat{\mathbf{k}} \cdot \nabla \omega = \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial y}$$

$$\therefore [\mathrm{B3}K0K]_e = \int_{\Omega_e} \left[\frac{\partial \{N\}}{\partial y} \{N\} \frac{\partial \{N\}^T}{\partial x} - \frac{\partial \{N\}}{\partial x} \{N\} \frac{\partial \{N\}^T}{\partial x} \right] \mathrm{d}x\mathrm{d}y$$

$$\Rightarrow [\mathrm{B3Y0X}\ ]_e - [\mathrm{B3X0Y}\ ]_e$$

Vorticity Robin BC generated via TS on $\partial \Omega^h$:

$$-\nabla^2 \psi - \omega = \frac{-\partial^2 \psi}{\partial s^2} - \frac{\partial^2 \psi}{\partial n^2} - \omega = 0 \Rightarrow \frac{\mathrm{d}^2 \psi}{\mathrm{d}n^2} = -\omega \ \Big|_{\partial \Omega_{\mathrm{wall}}}$$



TS:

$$\psi(\Delta n) = \psi_{\mathrm{w}} + \frac{\mathrm{d}\psi}{\mathrm{d}n}\Big|_{\mathrm{w}} \Delta n + \frac{\mathrm{d}^2 \psi}{\mathrm{d}n^2}\Big|_{\mathrm{w}} \frac{\Delta n^2}{2} + \frac{\mathrm{d}^3 \psi}{\mathrm{d}n^3}\Big|_{\mathrm{w}} \frac{\Delta n^3}{6} + O(\Delta n^4)$$

$$= \psi_{\mathrm{w}} + \mathrm{U}_{\mathrm{w}} \Delta n - \omega_{\mathrm{w}} \Delta n^2 / 2 - (\mathrm{d}\omega / \mathrm{d}n)_{\mathrm{w}} \Delta n^3 / 6$$

BC:

$$\ell(\omega) = \hat{\mathbf{n}} \cdot \nabla \omega - (3/\Delta n)\omega + (6/\Delta n^2)(\mathrm{U}_{\mathrm{w}}) - (6/\Delta n^3)(\psi_{\mathrm{w}+1} - \psi_{\mathrm{w}}) = 0$$

FM.21 Newton $\{FQ\}_e$ Template, (ω^h, ψ^h) GWS^h

GWS^h ⇒ global Newton statement

$$[\text{Jacobian}]\{\delta Q\}^{p+1} = -\{\text{F}Q\}^p \Leftrightarrow \text{S}_e\left([\text{JAC}]_e\right)\{\delta Q\}^{p+1} = -\text{S}_e\left(\{\text{F}Q\}_e\right)$$

Template pseudo code for $\{FQ\}_e$

$$\{\text{WS}(\cdot)\}_e \equiv (\text{const})\ (\text{avg})_e\{\text{dist}\}_e(\text{metric;det})_e[\text{matrix}]\{Q \text{ or data}\}_e$$

$$\begin{aligned}\{\text{FOMG}\}_e = &(\text{Re}^{-1})(\ )\{\ \}(-1)[\text{B}2KK]\{\text{OMG}\}\\ &+(\ )(\ )\{\text{PSI}\}(-1)[\text{B}3K0K]\{\text{OMG}\}\\ &+(-3/\Delta n, \text{Re}^{-1})(\ )\{\ \}(1)[\text{A}200]\{\text{OMG} - f(\text{U}, \Delta\psi\}\end{aligned}$$

$$\begin{aligned}\{\text{FPSI}\}_e = &(\ )(\ )\{\ \}(-1)[\text{B}2KK]\{\text{PSI}\}\\ &+(-1)(\ )\{\ \}(1)[\text{B}200]\{\text{OMG}\}\\ &+(\ )(\ )\{\ \}(1)[\text{A}200]\{\text{U}\}\end{aligned}$$

FM.22 Newton Jacobian Template, (ω^h, ψ^h) GWS^h

Newton jacobian formed via differentiation

$$[\mathrm{JAC}]_e \equiv \frac{\partial\{\mathrm{F}Q\}_e}{\partial\{Q\}_e} = \begin{bmatrix} [\mathrm{J\Omega\Omega}] & [\mathrm{J\Omega\psi}] \\ [\mathrm{J\psi\Omega}] & [\mathrm{J\psi\psi}] \end{bmatrix}_e$$

Jacobian template pseudo-code

$$[\mathrm{J\Omega\Omega}]_e \equiv \frac{\partial\{\mathrm{FOMG}\}_e}{\partial\{\mathrm{OMG}\}_e} = (\mathrm{Re}^{-1})(\ )\{\ \}(-1)[\mathrm{B2}KK][\ \ ] + (\ )(\ )\{\mathrm{PSI}\}(-1)[\mathrm{B3}K0K][\ \ ]$$

$$-(3/\Delta n, \mathrm{Re}^{-1})(\ )\{\ \}(1)[\mathrm{A200}][\ \ ]$$

$$[\mathrm{J\Omega\psi}]_e \equiv \frac{\partial\{\mathrm{FOMG}\}_e}{\partial\{\mathrm{PSI}\}_e} = (\ )(\ )\{\mathrm{OMG}\}(-1)[\mathrm{B3}K0K\mathrm{T}][\ \ ] - (6/\Delta n^3, \mathrm{Re}^{-1})(\ )\{\ \}(1)[\mathrm{A200}][\Delta\mathrm{PSIwall}]$$

$$[\mathrm{J\psi\Omega}]_e \equiv \frac{\partial\{\mathrm{FPSI}\}_e}{\partial\{\mathrm{OMG}\}_e} = (-1)(\ )\{\ \}(1)[\mathrm{B200}][\ \ ]$$

$$[\mathrm{J\psi\psi}]_e \equiv \frac{\partial\{\mathrm{FPS}I\}_e}{\partial\{\mathrm{PSI}\}_e} = (\ )(\ )\{\ \}(-1)[\mathrm{B2}KK][\ \ ]$$

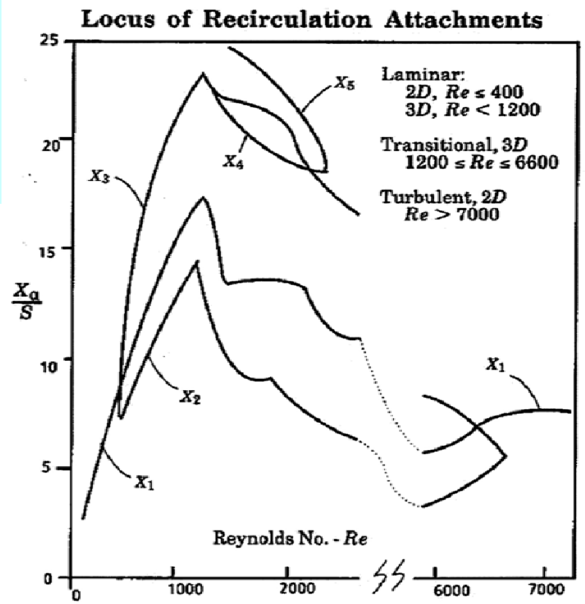
FM.23 Incompressible N-S, Step Wall Diffuser

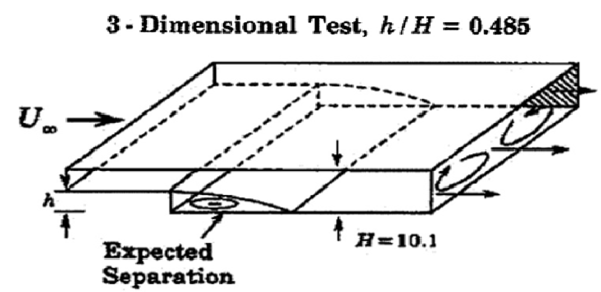
GWS^h + Newton ⇒ computable matrix statement

$$S_e\left(\begin{bmatrix} J\Omega\Omega & J\Omega\psi \\ J\psi\Omega & J\psi\psi \end{bmatrix}_e \begin{Bmatrix} \delta OMG \\ \delta PSI \end{Bmatrix}_e^{p+1} = -\begin{Bmatrix} FOMG \\ FPSI \end{Bmatrix}_e^{p}\right)$$

Step-well diffuser problem statement

primary separated flow region
caused by step change in $A(\mathbf{x})$
reattachment $x_1(\psi_w) = f\,(\mathrm{Re})$
multiple auxiliary separations in 3-D





Computer lab design problem

determine x_1/S of dividing Ψ_1 as $f\,(\mathrm{Re})$
for $100 \le \mathrm{Re} \le 600$