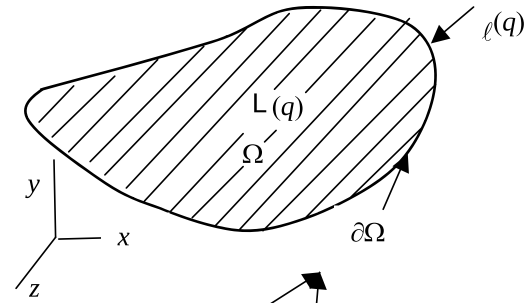
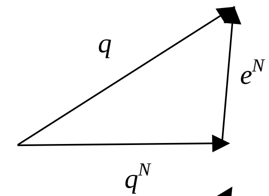
HC.1 FE Weak Statement Algorithm Steps



The (heat conduction) problem statement

$$\mathrm{L}(T) = 0 \ on\ \Omega + \text{BCs}$$

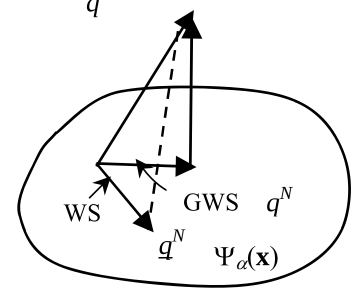
Approximate solution, with associated error



$$T^N(x) = \sum_{\alpha=1}^{N} \Psi_\alpha(x)\, Q_\alpha$$

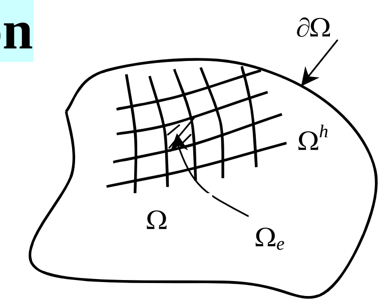
$$T(x) = T^N(x) + e^N(x)$$

Minimize the error via Galerkin weak statement



$$\text{GWS}^N \equiv \int_\Omega \Psi_\beta(x)\, \mathrm{L}(T^N)\, dx \equiv 0, \ 1 \le \beta \le N$$

Implement GWS^N via FE discrete approximation



$$\Omega \Rightarrow \Omega^h \equiv \cup_e \Omega_e$$

$$T^N \equiv T^h(x) \Rightarrow \cup_e T_e(x), \ \text{GWS}^N \Rightarrow \text{GWS}^h$$

Solve matrix statement

$\text{GWS}^h \Rightarrow [\text{Matrix}]\{Q\} = \{\text{b}\}$, hence evaluate error $e^{\boldsymbol{h}}(x)$

HC.2 An Example, Heat Conduction in a Slab

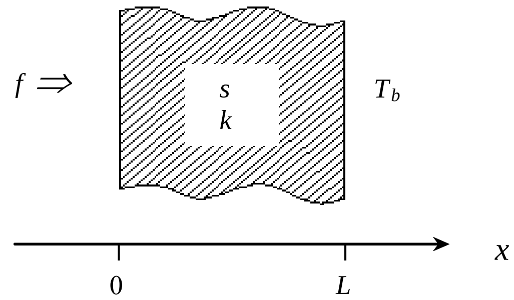
Example problem

$$\mathcal{L}(T) = -\frac{d}{dx}\left(k\frac{dT}{dx}\right) - s = 0, \qquad on \quad 0 < x < L$$

$$\ell(T) = -k\frac{dT}{dx} - f = 0 \qquad at \quad x = 0$$

$$T(L) = T_b \qquad at\ x = L$$

problem data



Analytical solution

$$T(x) = \frac{sL^2}{2k}\left[1 - \left(\frac{x}{L}\right)^2\right] + \frac{fL}{k}\left(1 - \frac{x}{L}\right) + T_b$$

Any approximate solution

$$T^N(x) = \sum_{\alpha=1}^{N} \Psi_\alpha(x)Q_\alpha = Q_1\Psi_1(x) + Q_2\Psi_2(x) + \ldots + Q_N\Psi_N(x)$$

For this simple problem, $T^N \Rightarrow T(x)$ for $N = 3$ via

$$Q_1 = \frac{sL^2}{2k},\ Q_2 = \frac{fL}{k},\ Q_3 = T_b;\ \Psi_1 = 1 - \left(\frac{x}{L}\right)^2,\ \Psi_2 = 1 - \left(\frac{x}{L}\right),\ \Psi_3 = 1$$

HC.3 Approximation, Constraint on Error

***Any* approximation**

$$T^N(x) = \sum_{\alpha=1}^{N} \Psi_\alpha(x) Q_\alpha$$

The *error* in T^N is e^N, recall

$$T(x) = T^N(x) + e^N(x)$$

No knowledge of e^N exists, however $\mathrm{L}(T^N) = -\mathrm{L}(e^N)$

$$\mathrm{L}\,(T^N) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(k\frac{\mathrm{d}T^N}{\mathrm{d}x}\right) - s \neq 0$$

Error minimized via Galerkin weak statement

$$\mathrm{GWS}^N \equiv \int \Psi_\beta(x)\,\mathrm{L}\,(T^N)\,\mathrm{d}x \equiv 0,\ 1 \leq \beta \leq N$$

HC.4 Galerkin Weak Statement, Minimum Error

Approximation

$$T^N(x) \equiv \sum_{\alpha=1}^{N} \Psi_\alpha(x) Q_\alpha$$

Galerkin weak statement

$$\mathrm{GWS}^N \equiv \int_\Omega \Psi_\beta(x)\left[-\frac{\mathrm{d}}{\mathrm{d}x}\left(k\frac{\mathrm{d}T^N}{\mathrm{d}x}\right) - s\right]\mathrm{d}x \equiv 0, \qquad \mathit{for} \quad 1 \le \beta \le N$$

Integrating by parts, substituting $T^N(x)$ and BC f_n

$$\mathrm{GWS}^N = \sum_{\alpha=1}^{N}\left(\int_\Omega \frac{\mathrm{d}\Psi_\beta}{\mathrm{d}x} k \frac{\mathrm{d}\Psi_\alpha}{\mathrm{d}x}\mathrm{d}x\right)Q_\alpha - \int_\Omega \Psi_\beta s \mathrm{d}x - k\frac{\mathrm{d}T^N}{\mathrm{d}x}\Psi_N\Big|_{x=\mathrm{L}} - f_n \Psi_1\Big|_{x=0} = 0$$

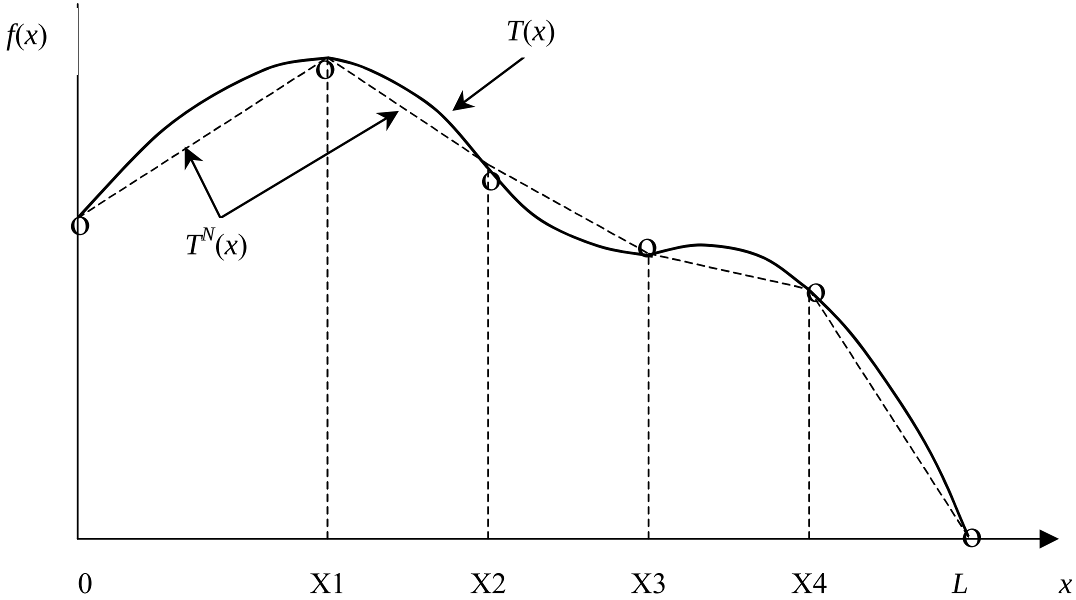
for $1 \le \beta \le N$, and heat flux BC is directly *embedded*

HC.5 Trial Functions, Interpolation

To complete the integrals in the GWS^N

$\Rightarrow$ must specify the trial space $\Psi_\alpha(x)$, $1 \le \alpha \le N$

Lagrange piecewise interpolation provides insight



Interpolation *error* can be adjusted by adding knots " o "

$\Rightarrow$ *nodes* of the FE *discretization* of $\Omega \Rightarrow \Omega^h = \cup_e \Omega_e$

HC.6 Discrete Approximation, Finite Element Basis

For N = 3 node FE mesh

$$T^N(x) = \sum_{\alpha=1}^{N=3} \Psi_\alpha(x) Q_\alpha = \Psi_1 Q_1 + \Psi_2 Q_2 + \Psi_3 Q_3$$

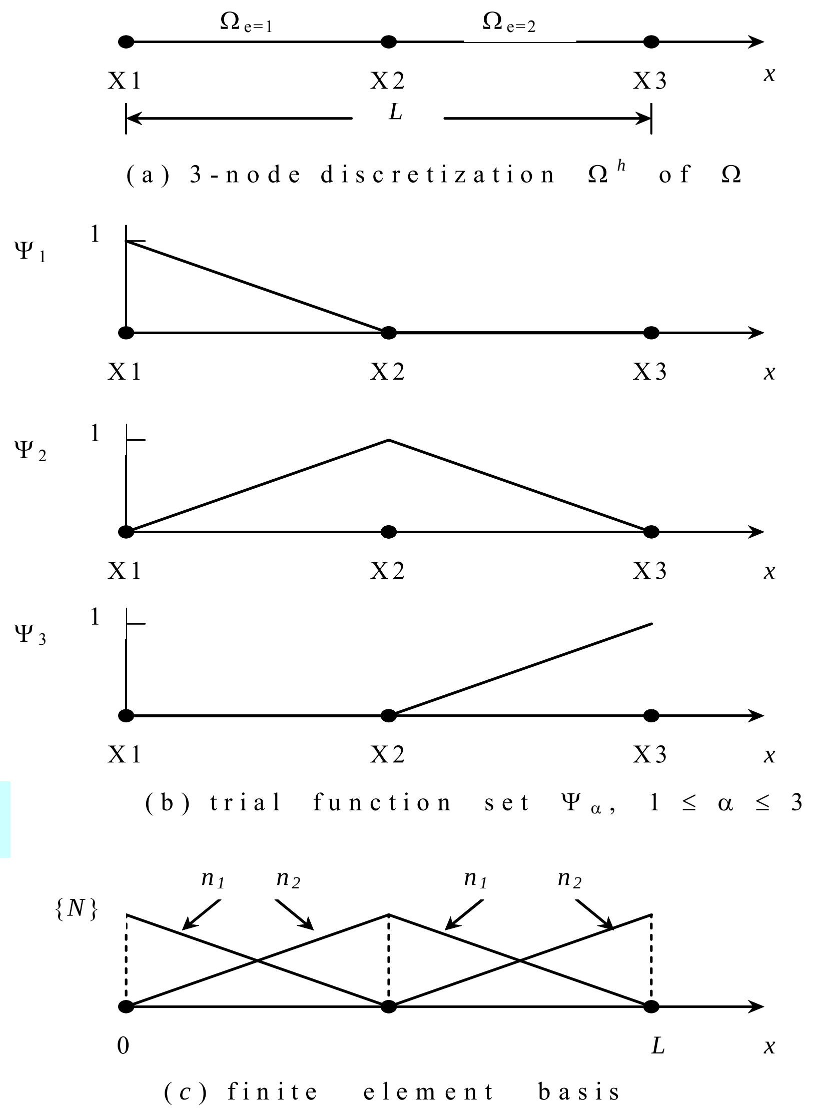
Global trial functions $\Psi_\alpha(x)$

$$\Psi_\alpha(x \Rightarrow \text{node } (\alpha)) \equiv 1$$
$$\Psi_\alpha(x \Rightarrow \text{node } (\beta \neq \alpha)) \equiv 0$$

Local finite element basis $\{N\}$

$$\{N\} = \left\{ \begin{matrix} n_1 = \dfrac{\text{XR} - x}{\text{XR} - \text{XL}} \\ n_2 = \dfrac{x - \text{XL}}{\text{XR} - \text{XL}} \end{matrix} \right\}_e$$

on every (!) element Ω_e



HC.7 Finite Element Matrix Library

GWS^N first term derivatives, subscripts ⇒ matrices

$$\int_{\Omega} \frac{d\psi_\beta}{dx}\frac{d\psi_\alpha}{dx} dx\, Q_\alpha \Rightarrow \int_{\Omega_e} \frac{d\{N\}}{dx}\frac{d\{N\}^T}{dx} dx\, \{Q\}_e, \quad and \quad \frac{dn_i}{dx} = \begin{Bmatrix} -1/\ell_e, i=1 \\ 1/\ell_e, i=2 \end{Bmatrix} = \frac{d\{N\}}{dx}$$

The integral of matrix products on Ω_e is

$$\int_{\Omega_e} \frac{d\{N\}}{dx} k \frac{d\{N\}^T}{dx} dx \{Q\}_e = k\int_0^{l_e} \frac{1}{l_e}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \frac{1}{l_e}\{-1 \quad 1\} dx\, \{Q\}_e$$

$$= \frac{k}{l_e^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_0^{l_e} dx\, \{Q\}_e = \frac{k}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\{Q\}_e$$

For the constant source term

$$\int_{\Omega_e} \{N\} s \; dx = s \int_0^{l_e} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} dx = \frac{s\, l_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Boundary conditions require no integration

HC.8 Finite Element Data Evaluations

The FE discrete implementation process yields

$$\mathrm{GWS}^N \Rightarrow \mathrm{GWS}^h = \sum_e \{\mathrm{WS}\}_e$$

$$\{\mathrm{WS}\}_e = \frac{k}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\{Q\}_e - \frac{s\ l_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} - k\frac{\mathrm{d}T}{\mathrm{d}x}\begin{Bmatrix} -\delta_{e1} \\ \delta_{eM} \end{Bmatrix}$$

δ_{ej} is a Kronecker delta on/off switch

Every contribution to $\{\mathrm{WS}\}_e$ involves a product

$$\{\mathrm{WS}\}_e = (\mathrm{data})_e \times [\ \mathrm{FE\ \ matrix}\ ]$$

$$for \quad e = 1: \quad \{\mathrm{WS}\}_1 = \frac{k}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\{Q\}_{e=1} - \frac{s l_1}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} - k\frac{\mathrm{d}T}{\mathrm{d}x}\begin{Bmatrix} -\delta_{11} \\ 0 \end{Bmatrix}$$

$$= \frac{k}{L/2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} Q1 \\ Q2 \end{Bmatrix} - \frac{sL/2}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} - \begin{Bmatrix} f \\ 0 \end{Bmatrix}$$

$$for \quad e = 2: \quad \{\mathrm{WS}\}_2 = \frac{k}{L/2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} Q2 \\ Q3 \end{Bmatrix} - \frac{sL}{4}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ F3 \end{Bmatrix}$$

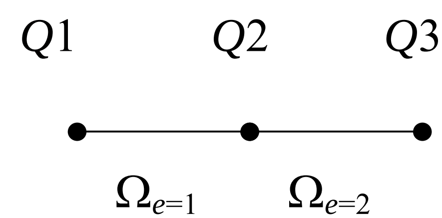
HC.9 FE Weak Statement Assembly over Ω^h

GWS^h is a matrix statement, i.e.,

$$\mathrm{GWS}^h = \sum_e \{\mathrm{WS}\}_e = [\mathrm{Matrix}]\{Q\} - \{\mathrm{b}\} = \{0\},$$

$$\{Q\} = \begin{Bmatrix} Q1 \\ Q2 \\ Q3 \end{Bmatrix}$$

[Matrix] and {b} involve a row summation process



$$[\mathrm{Matrix}] = \sum_{e=1}^{\mathrm{M}} [\mathrm{Matrix}]_e$$

$$= \frac{2k}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{2k}{L}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \frac{2k}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\{\mathrm{b}\} = \sum_{e=1}^{2} \{\mathrm{b}\}_e = \frac{sL}{4}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} + \begin{Bmatrix} f_n \\ 0 \\ 0 \end{Bmatrix} + \frac{sL}{4}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -F3 \end{Bmatrix}$$

assembly is universally valid for 1-D, 2-D and 3-D problems (!)

HC.10 Matrix Statement Solution, BCs

Assembling GWS^h over M = 2 FE domains Ω_e yields

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} Q1 \\ Q2 \\ Q3 \end{Bmatrix} = \frac{sL^2}{8k} \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix} + \frac{L}{2k} \begin{Bmatrix} f \\ 0 \\ -F3 \end{Bmatrix}$$

Substitute BC $Q3 = Tb$, move unknown flux $F3$ to left

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & L/2k \end{bmatrix} \begin{Bmatrix} Q1 \\ Q2 \\ F3 \end{Bmatrix} = \frac{sL^2}{8k} \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix} + \begin{Bmatrix} fL/2k \\ T_b \\ -T_b \end{Bmatrix}$$

As QM equations are *decoupled* from $F3$, Cramer's rule

$$\begin{Bmatrix} Q1 \\ Q2 \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \begin{Bmatrix} \frac{L}{2k}\left(\frac{sL}{4} + f_n\right) \\ \frac{sL^2}{4k} + T_b \end{Bmatrix} = \begin{Bmatrix} \frac{sL^2}{2k} + \frac{fL}{k} + T_b \\ \frac{3sL^2}{8k} + \frac{fL}{2k} + T_b \end{Bmatrix}$$

then solve for $F3 = sL + f$

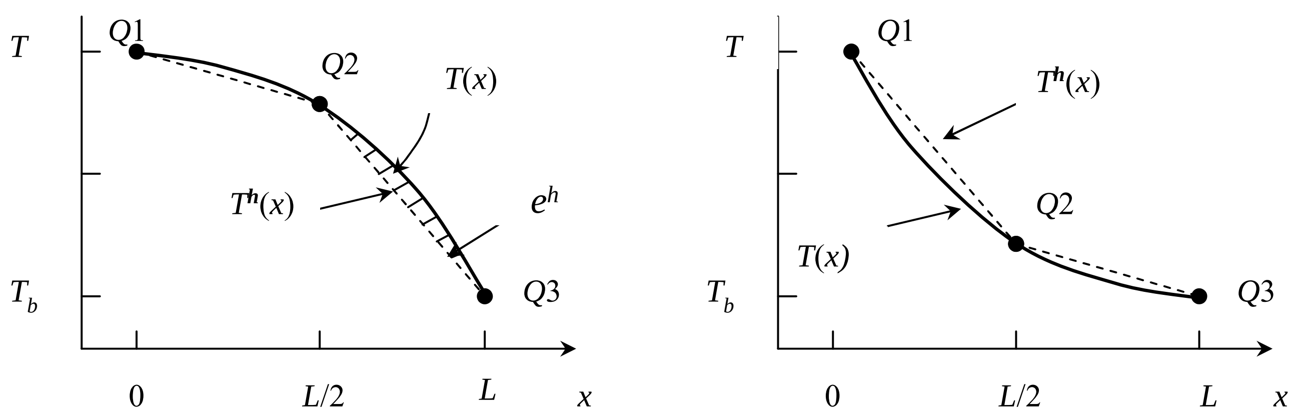
HC.11 Solution Accuracy, Error Distribution

The FE GWS^h nodal array $\{Q\}$ agrees with analytical solution

- this problem statement is very elementary
- concept of piecewise-continuous FE basis $\{N\}$ verified

T^h is still only an approximation!

- Taylor series *error* estimate: $e^h \approx O\left(\ell_e^2\right)$



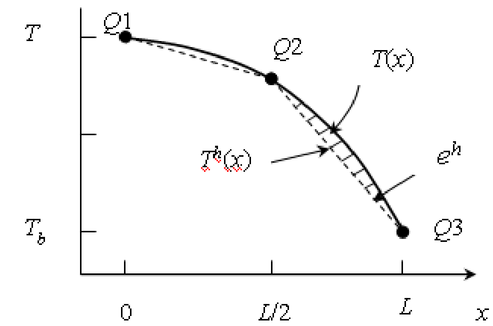
(a) Positive source term *s* (b) Negative source term *s*

HC.12 Boundary Heat Flux Computation

Boundary heat fluxes can be computed via

differentiating $T^h(x)$ at $x = L$

GWS^h matrix solution for $F3$



(a) Positive source terms

Differentiating T^h at $x = L$ yields

$$-k\frac{dT_{e=2}}{dx} = -\frac{k}{L/2}\left[T_b - \left(\frac{3sL^2}{8k} + \frac{f_n L}{2k} + T_b\right)\right] = \frac{3sL}{4} + f_n$$

$\Rightarrow$ *inexact* (same as FD result)

Solving for $F3$ from GWS^h matrix statement

$$F3 = -k\frac{dT^N}{dx}\bigg|_{x=L} = -\frac{k}{L/2}\left[T_b - \left(T_b + \frac{f_n L}{2k} + \frac{3sL^2}{8k}\right) - \frac{sL^2}{8k}\right] = sL + f_n$$

$\Rightarrow$ *exact*!

HC.13 Error Estimate, Quantization

Taylor series (TS) truncation error (TE) estimate

$$e^h \approx O(l_e^2) \equiv C\ell_e^2$$

Quantization of error uses uniform mesh refinement

$$\text{meshes} : \Omega^h,\ \Omega^{h/2},\ \Omega^{h/4}, \ldots$$

$$\text{solutions} : T^h + e^h = T = T^{h/2} + e^{h/2} = \ldots$$

$$\text{clear C}: e^h = 2^2\, e^{h/2}$$

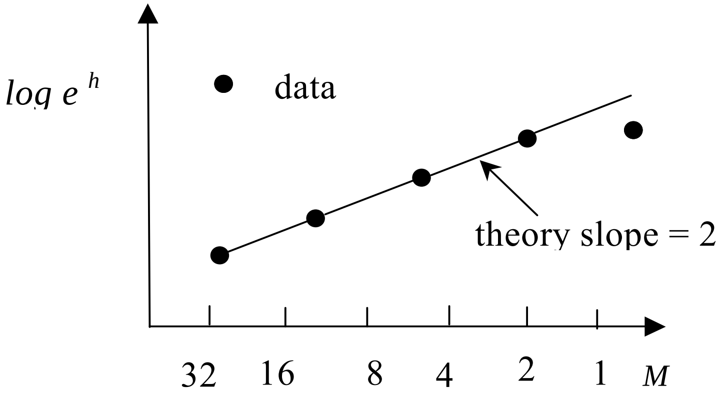
$$\text{hence} : T^{h/2} - T^h \equiv \Delta\, T^{h/2} = (2^2 - 1)\, e^{h/2} \Rightarrow e^{h/2} = \Delta T^{h/2}/3$$

Assessment of theory validity

$$slope \equiv \frac{\text{rise}}{\text{run}}$$

$$= \frac{\log(e^{h/M}) - \log e^{h/2M}}{\log(l_e) - \log(l_e/2)}$$

$$= \log(e^{h/M} / e^{h/2M}) / \log 2$$



HC.14 Error Estimation, Energy Norm

Improved error estimate uses entire solution via a "norm"

$$energy \quad norm \quad \equiv \left\| T \right\|_E^h \equiv \frac{1}{2}\int_\Omega k \frac{\mathrm{d}T^h}{\mathrm{d}x}\frac{\mathrm{d}T^h}{\mathrm{d}x}\mathrm{d}\tau \Rightarrow \frac{1}{2}\sum_e^M \{Q\}_e^T [\mathrm{DIFF}]_e \{Q\}_e$$

Uniform mesh refinement study

$$\left\| T^h \right\|_E + \left\| e^h \right\|_E = \left\| T \right\|_E = \left\| T^{h/2} \right\|_E + \left\| e^{h/2} \right\|_E = \ldots$$

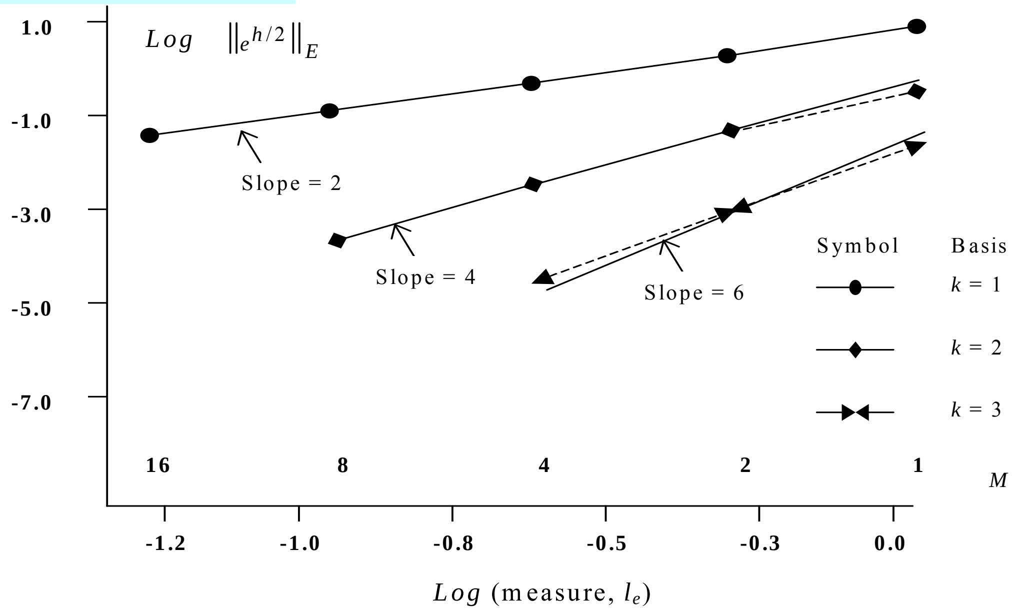
$$asymptotic \quad \text{convergence}: \left\| e^h \right\|_E \le C_k \ell_e^{2k} \left\| \text{data} \right\|_{L2}^2$$

error estimator

$$\| e^{h/2} \|_E = \frac{\Delta \| T^{h/2} \|_E}{2^{2k} - 1}$$

confirmation of theory

$$slope = \frac{\log \left[\| e^{h/M} \|_E / \| e^{h/2M} \|_E \right]}{\log 2}$$



HC.15 Error Quantization, $\{N_1\}$ FE Solution

Results from the computer lab exercise

Q_{max}

Mesh	M	l_e	$Q_1 \cdot 10^2$	$e^{h/2}$(est.)	slope
Ω^h	1	1.00000	2.50000		
$\Omega^{h/2}$	2	0.50000	2.50000	0.000000	
$\Omega^{h/4}$	4	0.25000	2.50000	0.000000	
$\Omega^{h/8}$	8	0.12500	2.50000	0.000000	
$\Omega^{h/16}$	16	0.06250	2.50000	0.000000	
$\Omega^{h/32}$	32	0.03125	2.50000	0.000000	
$\Omega^{h/64}$	64	0.01563	2.50000	0.000000	

$\left\| e^h \right\|_E$

Mesh	M	l_e	$\|\| T^{h/2} \|\|_E \, 10^4$	$\|\| e^{h/2} \|\|_E$(est)	slope
Ω^h	1	1.00000	1.12501		
$\Omega^{h/2}$	2	0.50000	1.15625	104.1667	
$\Omega^{h/4}$	4	0.25000	1.16406	26.04170	1.9999
$\Omega^{h/8}$	8	0.12500	1.16601	6.510417	2.0000
$\Omega^{h/16}$	16	0.06250	1.16650	1.627603	2.0000
$\Omega^{h/32}$	32	0.03125	1.16663	0.406902	1.9999
$\Omega^{h/64}$	64	0.01563	1.16663	0.101725	2.0000