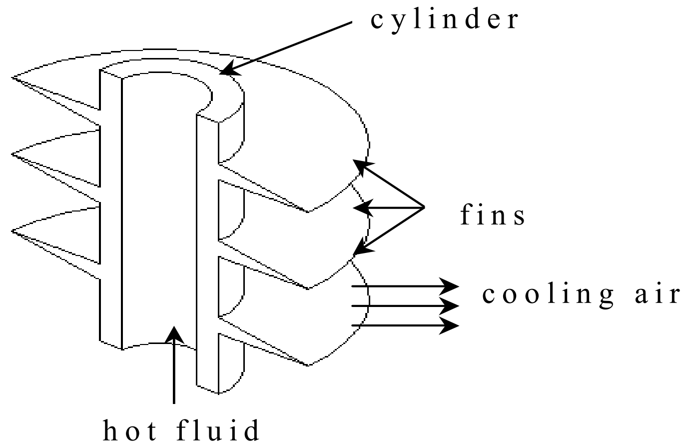
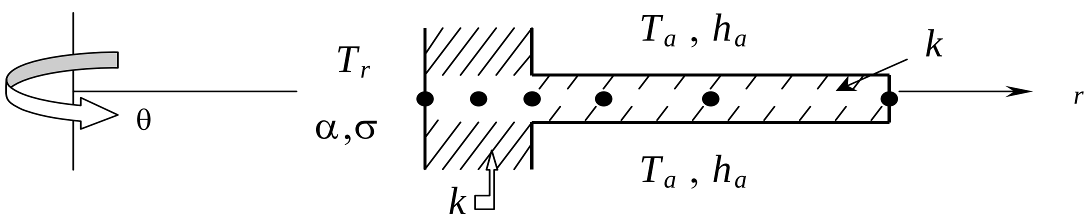
HT1.1 Heat Transfer, a Practical Problem on $n = 1$

IC engine design



analysis domain, generic fin



heat conduction is radial

DE:

$$\mathcal{L}(T) = -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[rk\,\frac{\mathrm{d}T}{\mathrm{d}r}\right] - s = 0, \qquad \text{on } \Omega \subset \Re^1$$

BCs:

$$\ell(T) = k\,\frac{\mathrm{d}T}{\mathrm{d}n} + h(T - T_a) + \alpha\sigma(T^4 - T_r^4) = 0, \quad \text{on } \partial\Omega_{h,\sigma}$$

HT1.2 Heat Transfer, $GWS^N \Rightarrow GWS^h$, $n = 1$

Approximation, GWS^N definition

$$T(r) \cong T^N(r) = \sum_{\alpha=1}^{N} \Psi_\alpha(r)\, Q_\alpha$$

$$GWS^N = \int_\Omega \Psi_\beta \mathrm{L}\left(T^N\right) d\tau = \int_\Omega \Psi_\beta \left[-\frac{1}{r}\frac{d}{dr}\left(kr \frac{dT^N}{dr} \right) - s(r) \right] r\,dr\,d\theta\,dz \equiv \{0\}, \quad 1 \le \beta \le N$$

$$= \int_\Omega \left[\frac{d\Psi_\beta}{dr} k \frac{dT^N}{dr} - \Psi_\beta s(r) \right] r\,dr\,d\theta\,dz + \int_{\partial\Omega_h} \Psi_\beta h(T^N - T_a)\, r\,dr\,d\theta\,dz$$

$$+ \int_{\partial\Omega_\sigma} \Psi_\beta \alpha\sigma (T^4 - T_r^4) r_c\, d\theta\,dz$$

FE implementation $\Rightarrow GWS^h$

$$T^N(r) \equiv T^h(r) = \cup_e T_e(r), \quad T_e(r) = \{N_k\}^T \{Q\}_e$$

$$GWS^h = \mathrm{S}_e \left[\int_{\Omega_e} \frac{d\{N\}}{dr} k_e \frac{d\{N\}^T}{dr} r\,dr\,d\theta\,dz \{Q\}_e - \int_{\Omega_e} \{N\} s_e\, r\,dr\,d\theta\,dz \right]$$

$$+ \left[\int_{\partial\Omega_e} \{N\} h(\{N\}^T \{Q\}_e - T_a)\, r\,dr\,d\theta + \int_{\partial\Omega_e} \{N\} \alpha\sigma (T_e^4 - T_r^4) r_c\, d\theta\,dz \right] = \{0\}$$

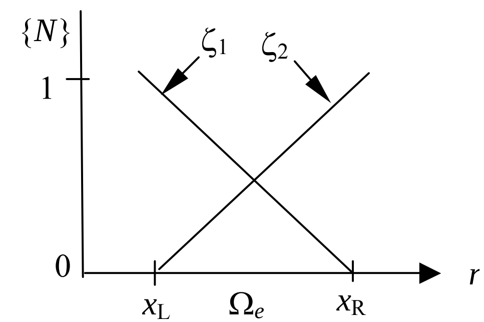
HT1.3 IC Engine Cylinder, FE GWS^h

FE basis

$$T(r) \approx T^N(r) \equiv T^h(r) = \cup_e T_e(r)$$

$$T_e(r) = \{N(\zeta)\}^T \{Q\}_e$$

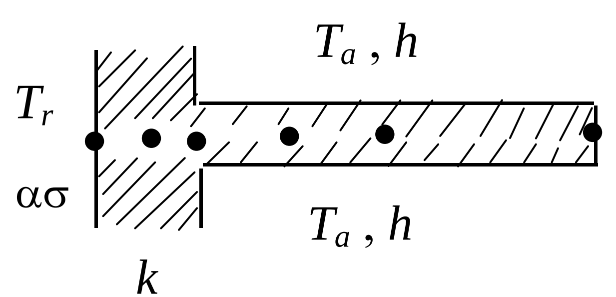
$$\{N_1\} = \begin{Bmatrix} \zeta_1 \\ \zeta_2 \end{Bmatrix} = \begin{Bmatrix} 1 - \bar{x}/l_e \\ \bar{x}/l_e \end{Bmatrix}$$



Galerkin weak statement

$$\mathrm{GWS}^N \Rightarrow \mathrm{GWS}^h = \mathrm{S}_e \{\mathrm{WS}\}_e = \{0\}$$

$$\{\mathrm{WS}\}_e = \int_{\Omega_e} \frac{\mathrm{d}\{N\}}{\mathrm{d}r} k_e \frac{\mathrm{d}\{N\}^T}{\mathrm{d}r} r \,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \{Q\}_e$$
$$+ \int_{\partial\Omega_e \cap \partial\Omega_\alpha} \{N\}\alpha\sigma(T_e^4 - \mathrm{TR}^4) r_c \,\mathrm{d}\theta\,\mathrm{d}z$$
$$+ \int_{\partial\Omega_e \cap \partial\Omega_h} \{N\} h \{N\}^T \{Q - \mathrm{TA}\}_e r \,\mathrm{d}r\,\mathrm{d}\theta$$



HT1.4 IC Engine Cylinder, $\{N_1\}$ FE Matrix Library

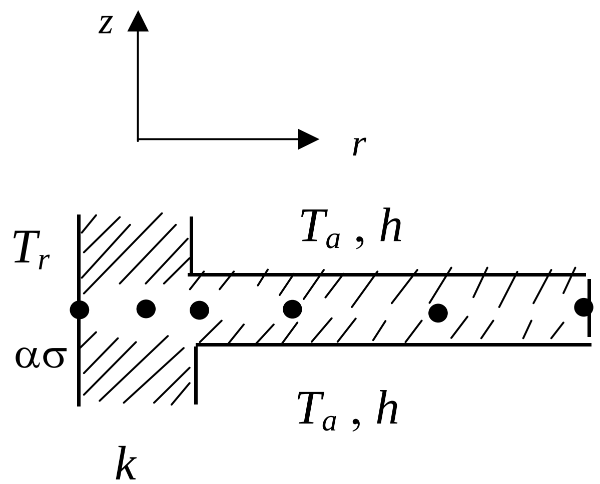
By terms in $\{\text{WS}\}_e$, linear FE basis

$$\int_{\Omega_e} (\cdot) \Rightarrow [\text{DIFF}]_e \{Q\}_e = \int_{\Omega_e} \frac{d\{N_1\}}{dr} k_e \frac{d\{N_1\}^T}{dr} r\, dr\, d\theta\, dz \{Q\}_e$$

$$= \frac{2\pi}{\ell_e^2} \Delta Z_e k \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \{-1, 1\} \int_r \{\text{R}\}_e^T \{N_1\} dr \{Q\}_e$$

$$= 2\pi \Delta Z_e k \ell_e^{-1} \{\text{R}\}_e^T \{1/2\} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \{Q\}_e$$

hence : $\{\text{WS}\}_e = (2\pi, \text{COND})(\Delta Z)_e \{\text{R}\}_e (-1)[\text{A}3011]\{Q\}_e$



Object-oriented *template* pseudo-code

$\{\text{WS}\}_e \Rightarrow$ (const) (avg)$_e$ $\{\text{dist}\}_e^T$ (metric)$_e$ [FE matrix] $\{Q \text{ or data}\}_e$

- (const) → scalar constant on *all* Ω_e
- (avg)$_e$ → scalar average on Ω_e
- $\{\text{dist}\}_e^T$ → distributed data on Ω_e
- (metric)$_e$ → exponent on l_e
- [FE matrix] → master element matrix
- $\{Q \text{ or data}\}_e$ → DOF or data on Ω_e

HT1.5 GWS^h FE Matrix Conventions

FE matrices result from integrations

$$[\text{Matrix}] = \int_{\Omega_e} (\text{conduction})\, d\tau, \quad \text{or } (\text{convection})\Big|_{\partial\Omega_e}$$

naming convention: [Matrix] ⇒ [M*bcc* … *d*]

M = denotes "Matrix," M ⇒ A for $n = 1$
b = integer, the number of FE bases $\{N\}$ in the integrand
c = Boolean index (0,1) repeated "*b*" times indicating each integrand basis $\{N\}$ (*no*, *yes*) differentiated
d = label for *any* needed identification

Examples from previous and next page

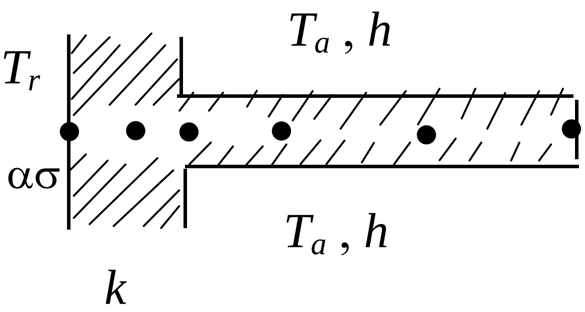
$$[\text{A3011L}] \equiv \ell_e \int_{\Omega_e} \{N_1\} \frac{d\{N_1\}}{dr} \frac{d\{N_1\}^T}{dr} dr \Rightarrow \{1/2\} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} & -\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \\ -\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} & \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \end{bmatrix}$$

$$[\text{A3000L}] \equiv \frac{1}{\ell_e} \int_{\Omega_e} \{N_1\}\{N_1\}\{N_1\}^T dr = \frac{1}{12} \begin{bmatrix} \begin{Bmatrix} 3 \\ 1 \end{Bmatrix} & \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \\ \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} & \begin{Bmatrix} 1 \\ 3 \end{Bmatrix} \end{bmatrix}$$

HT1.6 IC Engine Cylinder, $\{N_1\}$ FE Matrix Library

By terms in $\{\mathrm{WS}\}_e$

$$\int_{\partial\Omega_h} (\cdot) \Rightarrow [\mathrm{HBC}]_e \{Q - \mathrm{TA}\} = \int_{\partial\Omega_h} \{N_1\} h \{N_1\}^T \{Q - \mathrm{TA}\}_e r \, \mathrm{d}r \, \mathrm{d}\theta$$
$$= 2\pi h \{\mathrm{R}\}_e^T \int_{\Omega_e} \{N_1\}\{N_1\}\{N_1\}^T \mathrm{d}r \{\cdot\}_e$$



$$\{\mathrm{WS}\}_e = (2\pi, 2)(\mathrm{H})\{\mathrm{R}\}(1)[\mathrm{A3000}]\{Q\}$$
$$+(-2\pi, 2)(\mathrm{H})\{\mathrm{R}\}(1)[\mathrm{A3000}]\{\mathrm{TA}\}$$

$$\int_{\partial\Omega_r} (\cdot) \Rightarrow [\mathrm{RBC}]_e \{Q^4 - \mathrm{TR}^4\}_e = \int_{\partial\Omega_r} \{N_1\} \alpha\sigma \{N_1\}^T \{\cdot\}^4 \mathrm{R}_c \mathrm{d}\theta \, \mathrm{dz}$$
$$= 2\pi\alpha\sigma \mathrm{R}_c \Delta \mathrm{Z} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \{Q^4 - \mathrm{TR}^4\}_{e=1}$$

$$\{\mathrm{WS}\}_e = (2\pi, \alpha\sigma, \mathrm{RC})(\Delta\mathrm{Z})\{\quad\}(\quad)[\mathrm{ONE}]\{Q \exp 4\}$$
$$+(-2\pi, \alpha\sigma, \mathrm{RC})(\Delta\mathrm{Z})\{\quad\}(\quad)[\mathrm{ONE}]\{\mathrm{TR} \exp 4\}$$

HT1.7 IC Engine Cylinder, {WS}$_e$ Template

Axisymmetric heat transfer with radiation

DE + BCs

$$\mathcal{L}(T) = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(kr\frac{\mathrm{d}T}{\mathrm{d}r}\right) - s \;= 0$$

$$\ell(T) = k\mathrm{d}T/\mathrm{d}n + h\,(T - T_a) + \alpha\sigma\,(T^4 - T_r^4) = 0$$

$$\mathrm{GWS}^h \Rightarrow \mathrm{S}_e\{\mathrm{WS}_e\} = \{0\}$$

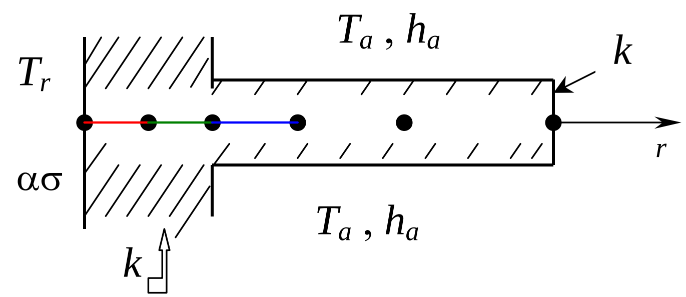
$$\{\mathrm{WS}\}_e = ([\mathrm{DIFF}]_e + [\mathrm{HBC}]_e + [\mathrm{RBC}(Q)]_e)\{Q\}_e - \{\mathrm{b}(T_r)\}_e$$

template:

$$\begin{aligned}
\{\mathrm{WS}\}_e &= (\mathrm{const})(\mathrm{avg})_e\{\mathrm{dist}\}_e^T(\mathrm{metric})_e[\mathrm{FE\ \ matrix}]\{Q\ \ or\ \ \mathrm{data}\}_e \\
&= (\mathrm{COND})(\Delta\mathrm{Z})\{\mathrm{R}\}(-1)[\mathrm{A3011}]\{Q\} \\
&\quad +(2)(\mathrm{H})\{\mathrm{R}\}(1)[\mathrm{A3000}]\{Q\} \\
&\quad +(-2)(\mathrm{H})\{\mathrm{R}\}(1)[\mathrm{A3000}]\{\mathrm{TA}\} \\
&\quad +(\mathrm{RC},\alpha\sigma)(\Delta\mathrm{Z})\{\,\}(\,)[\mathrm{ONE}]\{Q\ \ \exp 4\} \\
&\quad +(-\mathrm{RC},\alpha\sigma)(\Delta\mathrm{Z})\{\,\}(\,)[\mathrm{ONE}]\{\mathrm{TR}\ \ \exp 4\}
\end{aligned}$$

HT1.8 IC Engine Cylinder, GWS^h Assembled

Matrix statement: $\mathrm{GWS}^h \equiv \mathrm{S}_e\{\mathrm{WS}\}_e = \{0\}$



$$\{\mathrm{WS}\}_e = ([\mathrm{DIFF}]_e + [\mathrm{HBC}]_e + [\mathrm{RBC}(Q)]_e)\{Q\}_e - \{\mathrm{b}\}_e$$

M = 6 mesh, "diagonal" matrix form

$$\mathrm{S}_e\{\mathrm{WS}\}_e \Rightarrow \begin{bmatrix} 0 & k^1_{11} + Q1^3 & k^1_{12} \\ k^1_{21} & k^1_{22} + k^2_{11} & k^2_{12} \\ k^2_{21} & k^2_{22} + k^3_{11} + h^3_{11} & k^3_{12} + h^3_{12} \\ k^3_{21} + h^3_{21} & k^3_{22} + h^3_{22} + k^4_{11} + h^4_{11} & k^4_{12} + h^4_{12} \\ k^4_{21} + h^4_{21} & k^4_{22} + \ldots & . \\ . & . & . \\ . & k^7_{22} + h^7_{22} & 0 \end{bmatrix} \begin{Bmatrix} Q1 \\ Q2 \\ Q3 \\ Q4 \\ Q5 \\ Q6 \\ Q7 \end{Bmatrix} = \begin{Bmatrix} \mathrm{Tr}^4 \\ 0 \\ h\mathrm{Ta}_1{}^3 \\ h\mathrm{Ta}_2{}^3 + h\mathrm{Ta}\,{}^4_1 \\ h\mathrm{Ta}\,{}^4_2 + \ldots \\ . \\ h\mathrm{Ta}\,{}^7_2 \end{Bmatrix}$$

convective heat transfer acts
to augment radial conductivity via h
as local (negative) source via hT_a
radiation heat transfer makes GWS^h *very* non-linear

HT1.9 Non-linear GWS^h, n = 1, Newton Algorithm

Newton matrix iteration algorithm

$$[\text{Jacobian}]\{\delta Q\}^{p+1} = -\{\text{GWS}^h\}^p$$

$$\text{for } p \geq 0 : \{Q\}^{p+1} = \{Q\}^p + \{\delta Q\}^{p+1}, \textit{until} \max|\delta Q| < \varepsilon$$

Jacobian: $[\text{JAC}] \equiv \partial\{\text{GWS}^h\}/\partial\{Q\} \Rightarrow \text{S}_e[\text{JAC}]_e$

hence

$$\begin{aligned}[\text{JAC}]_e &= [\partial\{\text{WS}\}_e / \partial\{Q\}_e] \\ &= (\text{COND})(\Delta\text{Z})\{\text{R}\}(-1)[\text{A3011}][\ ] \\ &\quad +(2)(\text{H})\{\text{R}\}(1)[\text{A3000}][\ ] \\ &\quad +(4, \alpha\sigma, \text{RC})(\Delta\text{Z})\{\ \}(\ )[\text{ONE}][Q \ \ \exp 3]\end{aligned}$$

HT1.10 Non-linear GWS^h Template Pseudo-Code

GWS^h matrix solution essence

$$[\mathrm{JAC}]^P\{\delta Q\}^{P+1} = -\{\mathrm{FQ}\}^P, \quad \mathrm{GWS}^h \Rightarrow \{\mathrm{F}Q\} = \mathrm{S}_e\{\mathrm{WS}\}_e, \quad \{\mathrm{WS}\}_e = [\mathrm{DIFF}]_e + [\mathrm{BCs}]_e^p)\{Q\}_e - \{\mathrm{b}\}_e$$

$$\{Q\}^{P+1} = \{Q\}^P + \{\delta Q\}^{P+1}, \quad [\mathrm{JAC}] \equiv \partial\{\mathrm{F}Q\}/\partial\{Q\}, \quad [\mathrm{JAC}]_e = \frac{\partial\{\mathrm{WS}\}_e}{\partial\{Q\}_e}$$

Newton algorithm *template* pseudo-code

$$\{\mathrm{WS}\}_e = (\mathrm{const})(\mathrm{avg})_e\{\mathrm{dist}\}_e^T(\mathrm{metric})_e[\mathrm{FE\ \ matrix}]\{Q\ \ or\ \ \mathrm{data}\}_e$$

$$\begin{aligned}
\{\mathrm{F}Q\}_e = {} & (\mathrm{COND})(\Delta\mathrm{Z})\{\mathrm{R}\}(-1)[\mathrm{A3011}]\{Q\} \\
& +(2)(\mathrm{H})\{\mathrm{R}\}(1)[\mathrm{A3000}]\{Q\} \\
& +(-2)(\mathrm{H})\{\mathrm{R}\}(1)[\mathrm{A3000}]\{\mathrm{TA}\} \\
& +(\mathrm{RC},\alpha\sigma)(\Delta\mathrm{Z})\{\,\}(\,)[\mathrm{ONE}]\{Q\ \ \exp 4\} \\
& +(-\mathrm{RC},\alpha\sigma)(\Delta\mathrm{Z})\{\,\}(\,)[\mathrm{ONE}]\{\mathrm{TR}\ \ \exp 4\}
\end{aligned}$$

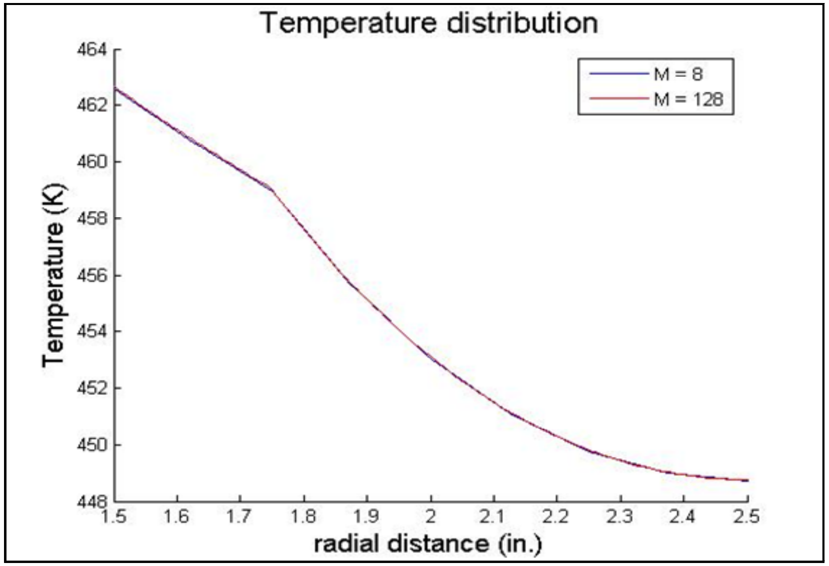
$$\begin{aligned}
[\mathrm{JAC}]_e = {} & (\mathrm{COND})(\Delta\mathrm{Z})\{\mathrm{R}\}(-1)[\mathrm{A3011}][\,] \\
& +(2)(\mathrm{H})\{\mathrm{R}\}(1)[\mathrm{A3000}][\,] \\
& +(4,\alpha\sigma,\mathrm{RC})(\Delta\mathrm{Z})\{\,\}(\,)[\mathrm{ONE}][Q\ \ \exp 3]
\end{aligned}$$

HT1.11 Heat Transfer, a Practical Problem on n = 1

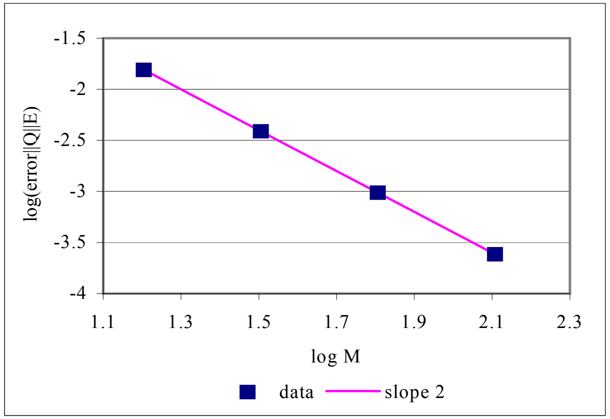
GWS^h performance, accuracy and convergence

$$\text{error estimate}: \left\| e^h \right\|_E \le C l_e^{2\gamma} \left\| \text{data} \right\|_{L2}^2 \quad \gamma = \min(k,\ r-1)$$

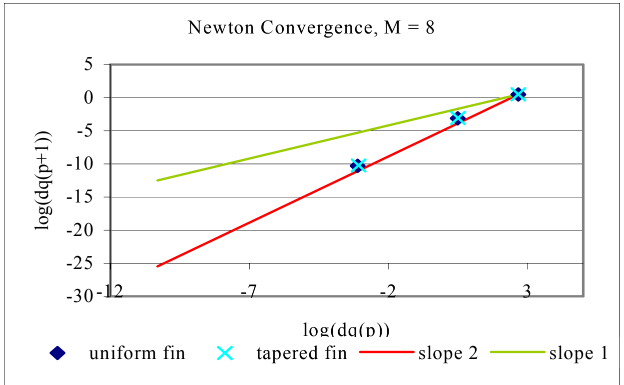
Temperature, uniform fin



Asymptotic convergence



Newton iterative convergence

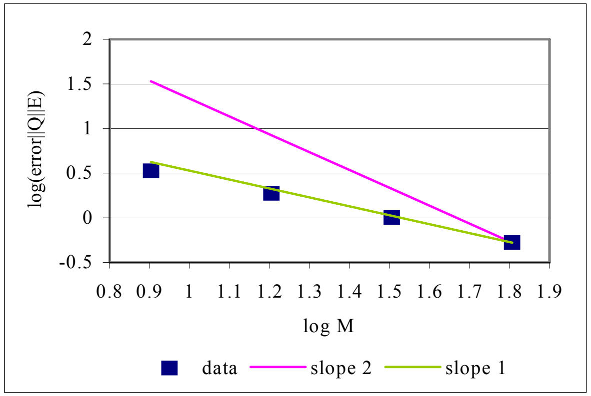


HT1.11A Heat Transfer, a Practical Problem on $n = 1$

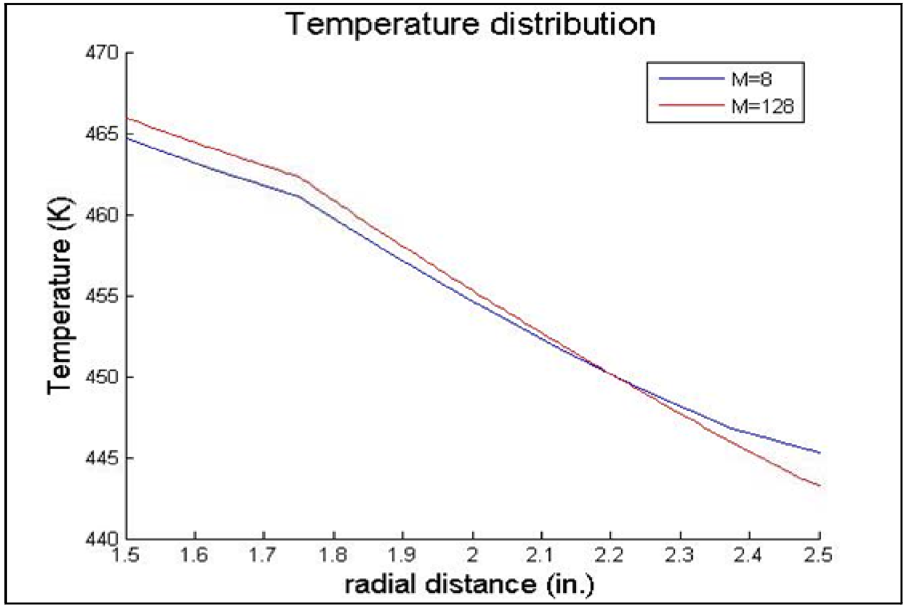
GWS^h performance, accuracy and convergence

$$\text{error estimate}: \left\| e^h \right\|_E \le C l_e^{2\gamma} \left\| \text{data} \right\|_{L2}^2 \quad \gamma = \min(k, r-1)$$

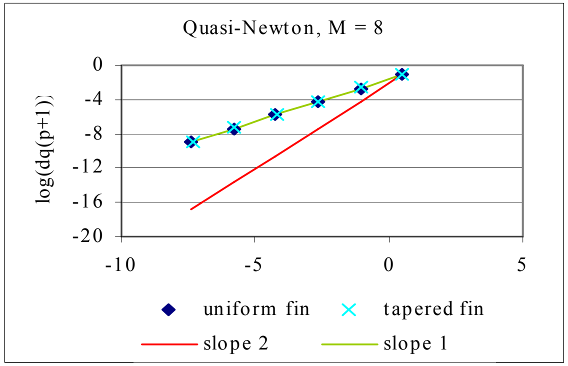
Asymptotic convergence



Temperature, tapered fin



quasi-Newton iterative convergence



HT1.12 D*E* GWS^h Summary, *n* = 1

Given D*E* + BC problem statement on *n* = 1

$$\mathrm{L}(q) = 0 \; on \; \Omega \subset \Re^1, \quad \ell(q) = 0 \quad on \; \partial\Omega$$

FE weak statement recipe

approximation: $T(x) \approx T^N(x) \equiv T^h(x) = \cup_e T_e(x)$

FE basis: $T_e(x) = \{N_k(\zeta)\}^T \{Q\}_e$

error extremization: $\mathrm{GWS}^N = \int_\Omega \Psi_\beta(x) \mathrm{L}\left(T^N\right) dx \equiv \{0\} \Rightarrow \mathrm{GWS}^h = \mathrm{S}_e\{\mathrm{WS}\}_e$

matrix statement: $\{\mathrm{WS}\}_e = \left([\mathrm{DIFF}]_e + [\mathrm{BCs}]_e\right)\{Q\}_e - \{\mathrm{b(data)}\}_e$

error estimation: $\left\|e^h\right\|_E \leq \mathrm{C}\, \ell_e^{2\gamma} \left\|\mathrm{data}\right\|_{\mathrm{L}2}^2, \quad \gamma \equiv \min(\, k+1-m, \, r-m)$

FE *template* pseudo-code

$$\{\mathrm{WS}\}_e = (\mathrm{const})\,(\mathrm{avg})_e \{\mathrm{dist}\}_e^T (\mathrm{metric})_e [\mathrm{FE\ matrix}] \{Q \; or \; \mathrm{data}\}_e$$