HT*n*.1 Heat Transfer on *n*-Dimensions

D*M*, D**P** & D*E* for *n*-D heat transfer ($\rho \approx$ constant)

$$\mathrm{D}M : \nabla \cdot \mathbf{V} = 0$$

$$\mathrm{D}\mathbf{P} : \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = \frac{\rho}{\rho_0}\mathbf{g} + \nabla \mathbf{T}$$

$$\mathrm{D}E : \frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T = s - \nabla \cdot \mathbf{q}$$

closure models

$$\mathbf{T} \Rightarrow -p/\rho_0 + \frac{\mu}{\rho_0} \cdot \nabla \mathbf{V}$$

$$\mathbf{q} \Rightarrow -k\nabla T$$

Heat transfer analyses characterized by non-D groups

potential temperature: $\Theta = (T - \mathrm{T}_{min})/\Delta \mathrm{T}$

closure terms:

$$\rho \mathbf{g}/\rho_0 \Rightarrow \mathrm{Gr}\,\mathrm{Re}^{-2}\,\Theta \hat{\mathbf{g}}$$

$$\mathbf{T} \Rightarrow \mathrm{Re}^{-1}$$

$$\mathbf{q} \Rightarrow (\mathrm{Re}\,\mathrm{Pr})^{-1}$$

$$\mathrm{BCs} \Rightarrow \mathrm{Nu}$$

Grashoff : $\mathrm{Gr} = \rho_0^2 \beta g \Delta \mathrm{T} \mathrm{L}^3 / \mu^2$

Reynolds : $\mathrm{Re} = \rho_0 \mathrm{U} \mathrm{L} / \mu$

Prandtl : $\mathrm{Pr} = c_p \mu / k$

Nusselt : $\mathrm{Nu} = h\mathrm{L}/k$

Rayleigh : $\mathrm{Ra} = \dfrac{\beta g k \Delta \mathrm{T}}{\rho c_p \mu \mathrm{U}^2}$

HTn.2 Heat Transfer GWS^h, n-Dimensions

DE, DP non-dimensionalized, n-dimensions

$$\text{D}E : \mathcal{L}(\Theta) = \frac{\partial \Theta}{\partial t} + \mathbf{u}\cdot\nabla\Theta - \frac{1}{\text{RePr}}\nabla^2\Theta - s = 0 \quad , \quad \text{on } \Omega \times t$$

$$\ell(\Theta) = \nabla\Theta\cdot\mathbf{n} + \text{Nu}\,(\Theta - \Theta_{ref}) + f_n = 0 \quad , \quad \text{on } \partial\Omega \times t$$

$$\Theta(\mathbf{x}_b, t) = \Theta_b(\mathbf{x}_b, t) \quad , \quad \text{on } \partial\Omega_b \times t$$

$$\Theta(\mathbf{x}, t_0) = \Theta_0(\mathbf{x}) \quad , \quad \text{on } \Omega \cup \partial\Omega \times t_0$$

$$\text{D}\mathbf{P} : \mathcal{L}(\mathbf{u}) = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P - \frac{1}{\text{Re}}\nabla^2\mathbf{u} + \frac{\text{Gr}}{\text{Re}^2}\Theta\mathbf{g} = 0$$

GWS^h finite element algorithm steps

approximation:

$$\Theta(\mathbf{x},t) \approx \Theta^N(\mathbf{x},t) \equiv \sum_{\alpha=1}^{N} \Psi_\alpha(\mathbf{x}) Q_\alpha(t)$$

extremize error:

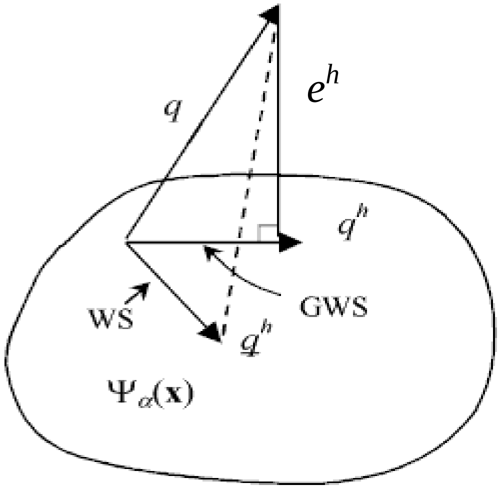
$$\text{GWS}^N \equiv \int_\Omega \Psi_\beta(\mathbf{x}) \mathcal{L}(\Theta^N)\, d\tau \equiv 0$$

FE discretization:

$$\Omega \cup \partial\Omega \approx \Omega^h \cup \partial\Omega^h$$

$$\Theta^N(\mathbf{x},t) \equiv \cup_e \Theta_e(\mathbf{x},t)$$

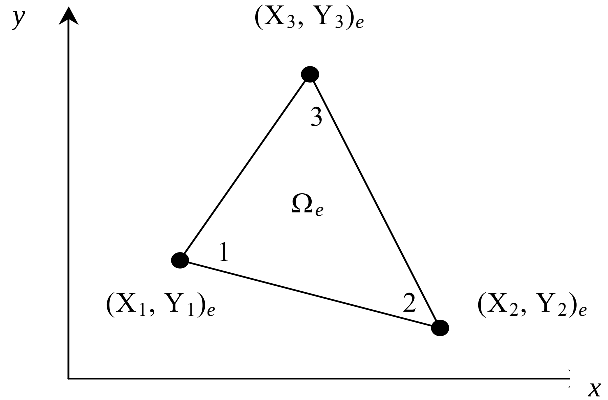
$$\Theta_e(\mathbf{x},t) = \{N_k(\zeta,\eta)\}^T \{Q(t)\}_e$$



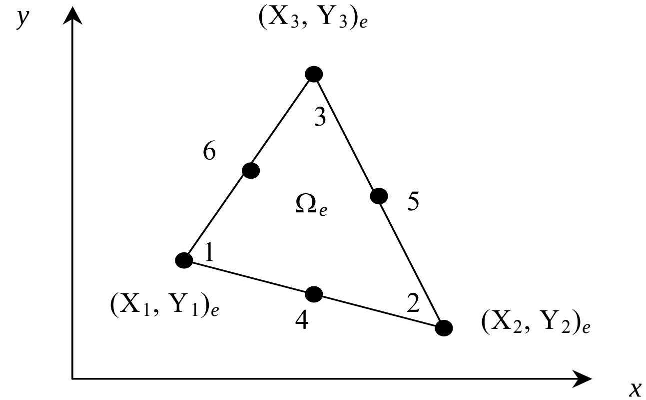
HT*n*.3 *n*-Dimensional Finite Elements

Natural coordinate family $\{N_k(\zeta)\}$, $k = 1, 2$

$n = 2$:

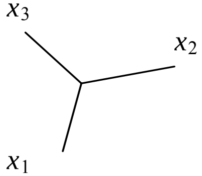


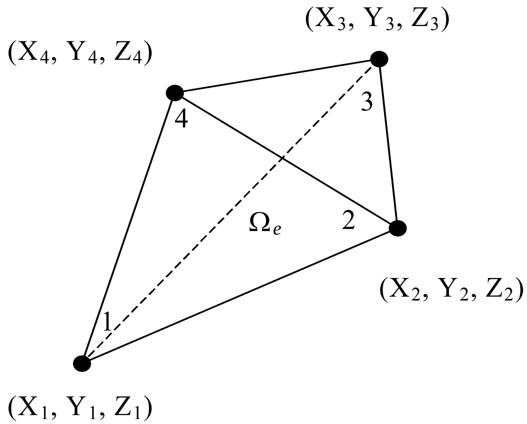
triangular element Ω_e



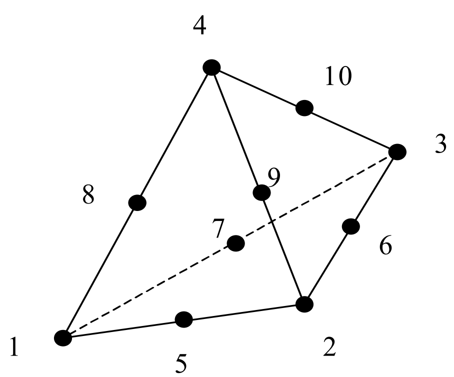
quadratic triangle

$n = 3$:





tetrahedron element Ω_e

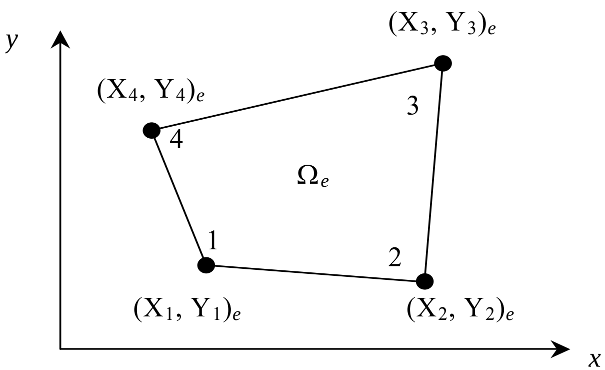


quadratic planar-face tetrahedron

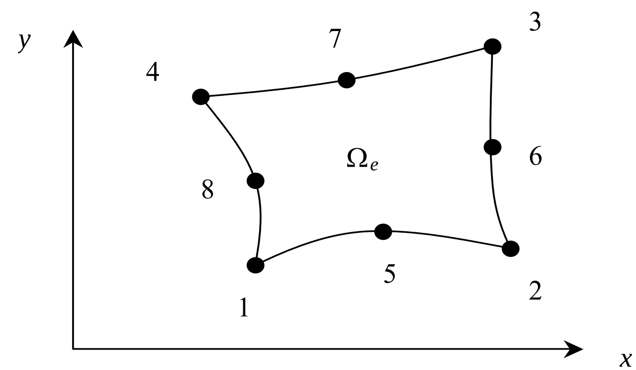
HT*n*.4 *n*-Dimensional Finite Elements

Tensor product family $\{N_k(\eta)\}$, $k = 1, 2$

$n = 2$:

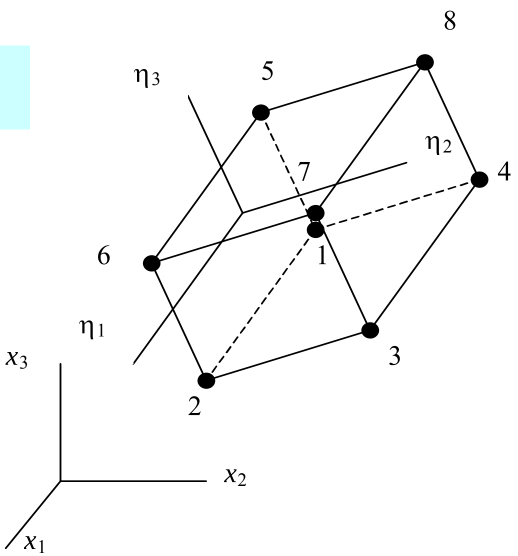


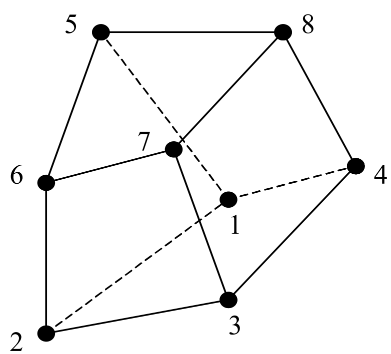
straight-sided



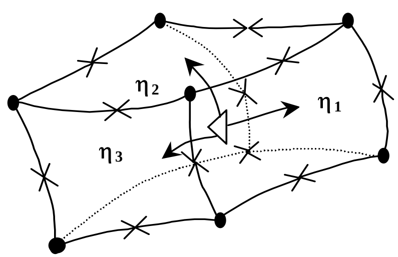
curve-sided

$n = 3$:





planar-faced

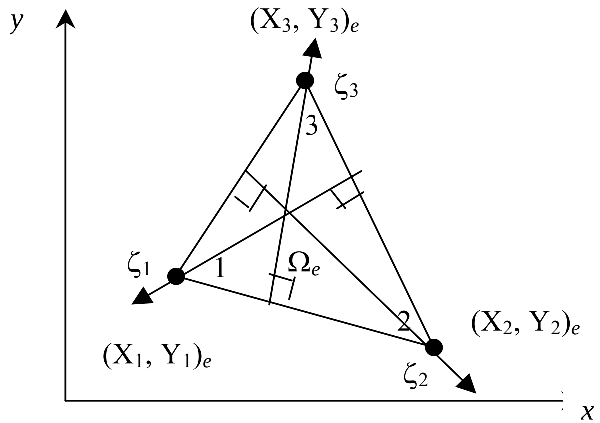


curved - faced

HT*n*.5 *n*-Dimensional Finite Element Basis

Natural coordinate basis, $\Theta_e(\mathbf{x},t) = \{N_1(\zeta)\}^T\{Q(t)\}_e$

$n = 2$:

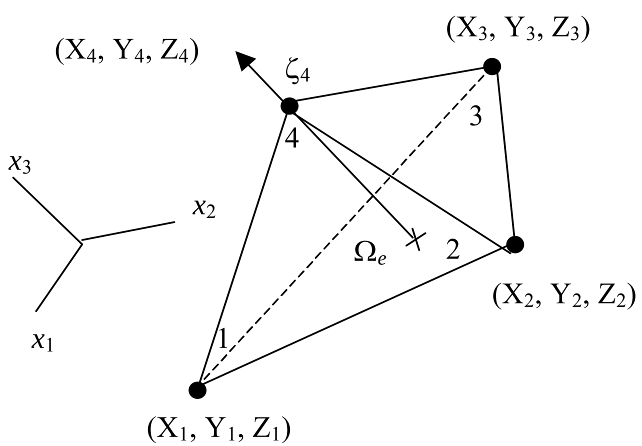


$$\{N_1(\zeta)\}^T \Rightarrow \{\zeta_1, \zeta_2\} \quad , n = 1$$
$$\Rightarrow \{\zeta_1, \zeta_2, \zeta_3\} \quad , n = 2$$
$$\Rightarrow \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\} \quad , n = 3$$

ζ_α system linearly dependent

$$\sum_{\alpha}^{n+1} \zeta_\alpha \equiv 1$$

$n = 3$:



GWS^h integrals easy to evaluate

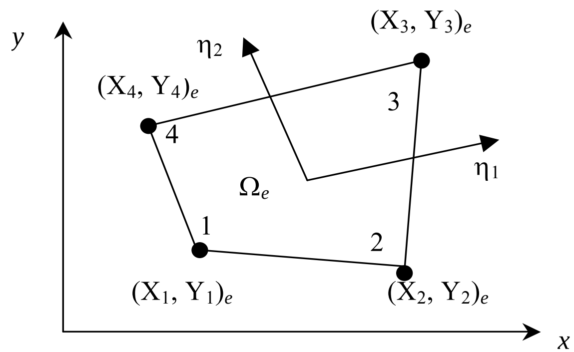
$$\int_{\Omega_e} \zeta_1^p \zeta_2^q \zeta_3^r \zeta_4^s = \det{}_e \frac{p!q!r!s!}{(n+p+q+r+s)!}$$

$\det_e \propto A_e, V_e$

HTn.6 n-Dimensional Finite Element Basis

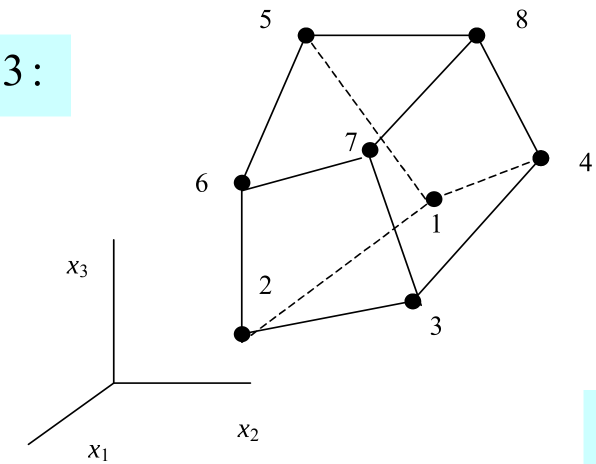
Tensor product basis, $\Theta_e(\mathbf{x},t) = \{N_1^+(\eta)\}^T\{Q(t)\}_e$

$n = 2$:



$$\{N_1^+(\eta_i)\} = \frac{1}{8}\begin{Bmatrix} (1-\eta_1)(1-\eta_2)(1-\eta_3) \\ (1+\eta_1)(1-\eta_2)(1-\eta_3) \\ (1+\eta_1)(1+\eta_2)(1-\eta_3) \\ (1-\eta_1)(1+\eta_2)(1-\eta_3) \\ (1-\eta_1)(1-\eta_2)(1+\eta_3) \\ (1+\eta_1)(1-\eta_2)(1+\eta_3) \\ (1+\eta_1)(1+\eta_2)(1+\eta_3) \\ (1-\eta_1)(1+\eta_2)(1+\eta_3) \end{Bmatrix}$$

$n = 3$:



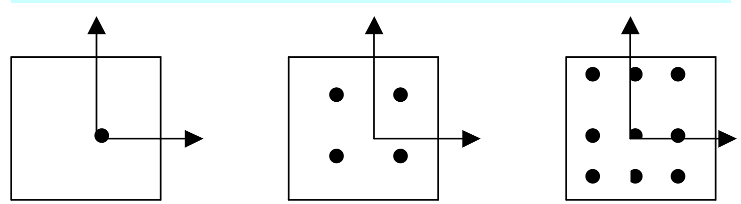
η_i system non-orthogonal, linearly *in*dependent

GWS^h integrals require numerical approximation

$$(\mathrm{diff}_{\alpha\beta})_e \equiv \iiint_{\hat{\Omega}_e} k_e \frac{\partial N_\alpha}{\partial \eta_j}\frac{\partial N_\beta}{\partial \eta_k}(\mathrm{eta}_{ji}\mathrm{eta}_{ki})_e \det{}_e^{-1}\, d\eta$$

$$\cong \sum_{i=1}^{n}\sum_{p=1}^{P}\sum_{q=1}^{Q}\sum_{r=1}^{R} H_p H_q H_r \quad \text{integrand } (\eta_j \Rightarrow \eta_i^{p,q,r})$$

Gauss quadrature:



P = 1 = Q P = 2 = Q P = 3 = Q

HTn.7 DE for Steady Heat Transfer, n-D GWS^h

$$\mathrm{GWS}^N = \int_\Omega \Psi_\beta(\mathbf{x}) \mathrm{L}(\Theta^N) d\tau = \{0\}$$

$$= \int_\Omega \Psi_\beta(\mathrm{x})[\mathbf{u}\cdot\nabla\Theta^N - (\mathrm{Re\,Pr})^{-1}\nabla\cdot\nabla\Theta^N - s]d\tau$$

$$= \int_\Omega [(\mathrm{RePr})^{-1}\nabla\Psi_\beta(\mathbf{x})\cdot\nabla\Theta^N + \Psi_\beta(\mathbf{u}\cdot\nabla\Theta^N - s)]\,d\tau - \oint_{\partial\Omega}\Psi_\beta(x)(\mathrm{RePr})^{-1}\nabla\Theta^N\cdot\hat{\mathbf{n}}\,d\sigma$$

$$= \int_\Omega [(\mathrm{RePr})^{-1}\nabla\Psi_\beta(\mathbf{x})\cdot\nabla\Theta^N + \Psi_\beta(\mathbf{u}\cdot\nabla\Theta^N - s)]\,d\tau + \int_{\partial\Omega_e\cap\partial\Omega}\Psi_\beta(\mathbf{x})\frac{1}{\mathrm{RePr}}\left[\mathrm{Nu}\left(\Theta^N - \Theta_r\right) + f_n\right]d\sigma$$

FE discrete implementation:

$$\Psi_\beta(\mathbf{x}) \Rightarrow \{N(\cdot)\}, \quad \int_\Omega(\cdot) \Rightarrow \int_{\Omega_e}(\cdot)$$

$$\mathrm{GWS}^h = \mathrm{S}_e\,\{\mathrm{WS}\}_e = \{0\}$$

$$\{\mathrm{WS}\}_e = ([\mathrm{DIFF}]_e + [\mathrm{UVEL}]_e + [\mathrm{HBC}]_e)\{Q\}_e - \{\mathrm{b}(s, \Theta_r, f_n)$$

$$\mathrm{S}_e\{\mathrm{WS}\}_e \Rightarrow [\mathrm{Matrix}\,(\mathbf{U}, \mathrm{Re}, \mathrm{Gr}, \mathrm{Pr}, \mathrm{Nu})]\{Q\} = \{\mathrm{b}\,(s, \Theta_r, f_n)\}$$

HT*n*.8 Summary, *n*-D GWS^h Essence for D*E*

FE discrete implementation GWS^h for steady D*E*

"recipe" $\Rightarrow$ analytical transformation of PDE plus BCs to algebraic (computable) form

analogous $\Rightarrow$ transformation methods for linear PDEs

solution^h $\Rightarrow$ parametric function of Re, Gr, Pr, Nu and *data*

error^h $\Rightarrow$ controllable via Ω^h and $\{N_k(\zeta,\eta)\}$ selections

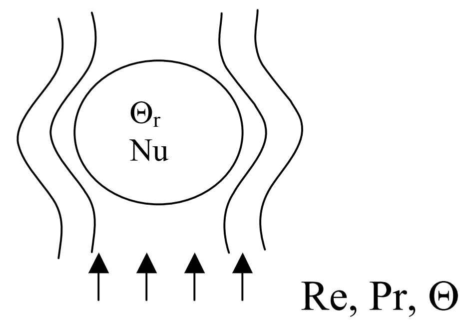
$$\left\| e^h \right\|_E \le C h^{2\gamma} \left\| \mathrm{data} \right\|_{L2}^2$$

$$\gamma \equiv \min(k+1-m,\, r-m),\, m=1$$

HT*n*.9 Natural & Mixed Convection, Heated Cylinder

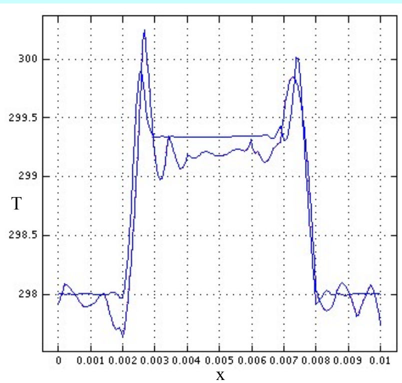
Computer Lab 4: cooling of a horizontal heated cylinder

data: u_{onset} (Re), Gr, Pr, diameter, Θ_{ref}
solution: Θ^h (x, y) as f (Re, Gr, Pr)
interpretation: modes of heat transfer

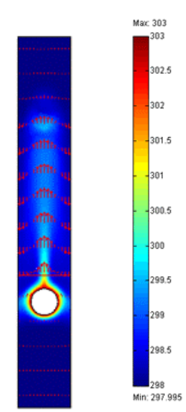


Natural, mixed & forced convection

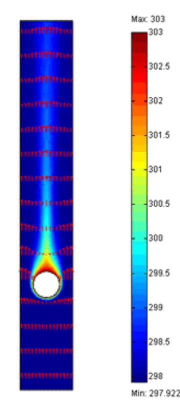
for Pr fixed, depends on Gr/Re^2



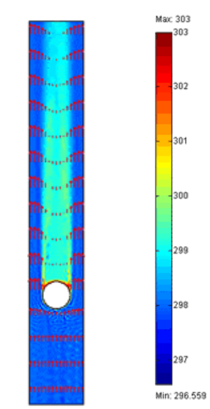
mesh refinement
for forced convection



natural convection



mixed convection



forced convection