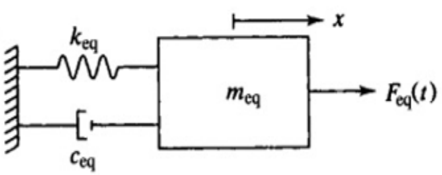
MV1.1 Mechanical Vibrations, Conservation Principles

Lagrangian viewpoint, mechanical system on n = 1

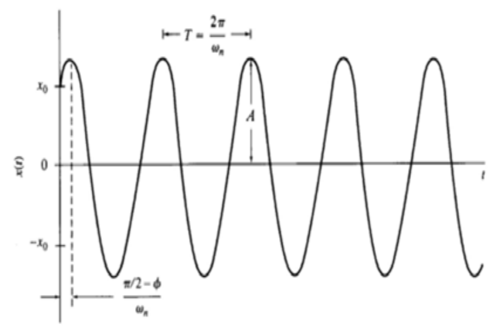
$$\mathrm{d}M = 0 = \mathrm{d}E$$
$$\mathrm{d}\mathbf{P} = \Sigma\mathbf{F} = m\mathbf{a}$$
$$\mathbf{r}\times \mathrm{d}\mathbf{P}\cdot\hat{\mathbf{k}} = \Sigma M_c = \mathrm{I}_c\alpha$$



$n = 1$ application, ICs $\Rightarrow x_0$, $\dot{x}_0$

free vibrations: $\mathrm{d}\mathbf{P} \Rightarrow m\ddot{x} + kx = 0$

solution: $x(t) = x_0\cos\omega t + \dot{x}_0\sin\omega t, \quad \omega = \sqrt{k/m}$



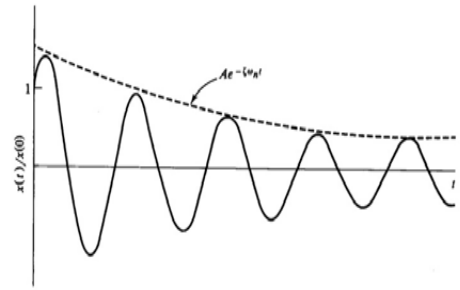
damped vibrations: $\mathrm{d}\mathbf{P} \Rightarrow \ddot{x} + (c/m)\dot{x} + (k/m)x = 0$

solution:

$$x(t) = \mathrm{B}e^{\alpha t}$$

$$\alpha_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$c_c = 2\sqrt{km} = \text{critical damping}$$



MV1.2 Mechanical Vibrations, Conservation Principles

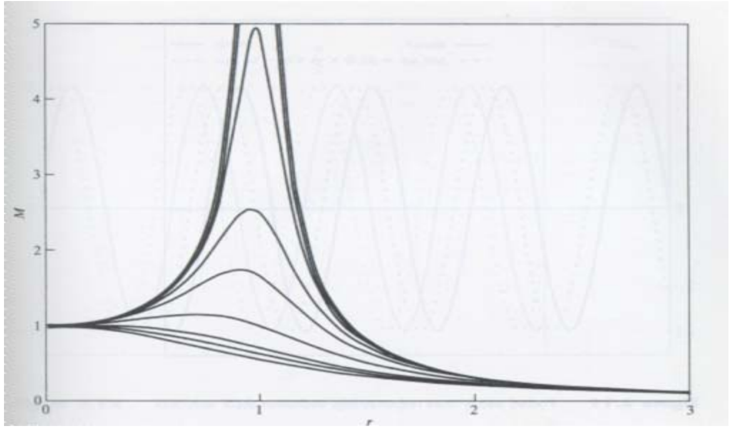
Lagrangian viewpoint, *n* = 1 applications continued

harmonic forcing: $\mathrm{d}\mathbf{P} \Rightarrow \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = m^{-1}F_0 \sin(\omega t + \varphi)$

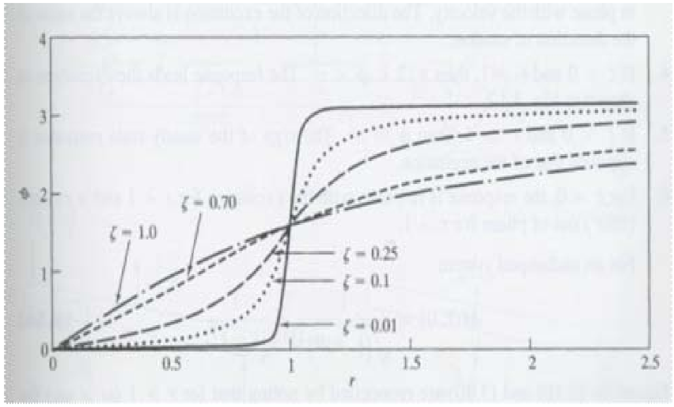
damping ratio: $\zeta \equiv c / 2\sqrt{km}$

solution: $x(t) = e^{-\zeta\omega_n t}\left[\mathrm{C}_1 \cos\left(\omega_n\sqrt{1-\zeta^2 t}\right) + \mathrm{C}_2 \sin\left(\omega_n\sqrt{1-\zeta^2 t}\right)\right]$

magnification factor: $\mathrm{M} = \omega_n^2\left((\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2\right)^{-1/2}$



magnification factor versus frequency ratio for damping ratios



phase angle versus frequency ratio for damping ratios

MV1.3 Mechanical Vibrations, Single DOF System

Transient vibrations

$$\mathbf{dP}: \ \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = F/m, \ \omega_n = \sqrt{k/m}, \quad \zeta = c/2\sqrt{km}$$

numerical solution

dual ODEs:

$$\dot{x} \equiv v$$
$$\dot{v} = F/m - 2\zeta\omega_n v - \omega_n^2 x$$

Taylor series:

$$\left.\begin{array}{l} x(t+\Delta t) = x(t) + \Delta t\dot{x} \Rightarrow v\,(t+\theta\Delta t) + O\,(\Delta t^{f(\theta)}) \\ v(t+\Delta t) = v(t) + \Delta t\dot{v} \Rightarrow (F/m - 2\zeta\omega_n v + \omega_n^2 x)_{t+\theta\Delta t} \end{array}\right\} \text{for } 0 \le \theta \le 1$$

note: yields matrix system for $\theta \neq 0$

Runge Kutta:

$$\dot{q} = f(q, t)$$
$$q_{t+1} = q_t + \sum_{j=1}^{n} a_j k_j$$

$O(\Delta t^4)$:

$$q_{t+1} = q_t + \frac{1}{6}(k_1 + k_2 + k_3 + k_4)$$

$$k_1 \equiv \Delta t\ f(q,t)$$
$$k_2 \equiv \Delta t\ f\left(q + \frac{1}{2}k_1,\ t_{n+1/2}\right)$$
$$\vdots$$
$$k_n$$

MV1.4 Mechanical Vibrations, Multiple DOF Systems

Lagrange's equation

dE: $L \equiv T - V =$ (kinetic energy - potential energy)$_{system}$

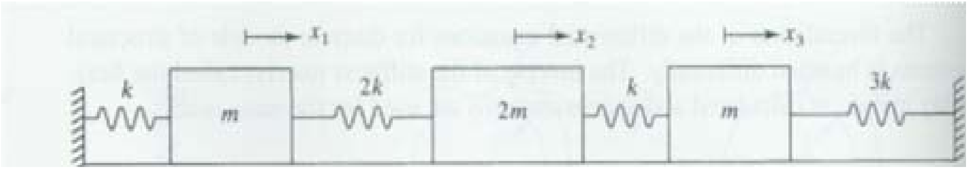
$= f(x_1, x_2, x_3 \cdots x_n;\ \dot{x}_1, \dot{x}_2, \dot{x}_3 \cdots \dot{x}_n)$

extremize L w.r.t. $2n$ DOF, yields Euler-Lagrange equation

d**P**:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0, \quad 1 \le i \le n$$

example:



$$T = m/2\left(\dot{x}_1^2 + 2\dot{x}_2^2 + \dot{x}_3^2\right)$$

$$V = k/2\left(x_1^2 + 2\,(x_2 - x_1)^2 + (x_3 - x_2)^2 + 3x_3^2\right)$$

MV1.5 Mechanical Vibrations, Multiple DOF Systems

Lagrangian, $L = T - V$:

$$T = m/2\left(\dot{x}_1^2 + 2\dot{x}_2^2 + \dot{x}_3^2\right)$$
$$V = k/2\left(x_1^2 + 2\,(x_2 - x_1)^2 + (x_3 - x_2)^2 + 3x_3^2\right)$$

extremization $\Rightarrow$ $\mathrm{d}\mathbf{P}_i$

$$\mathrm{d}\mathbf{P}_1 : \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0, \quad 1 \le i \le n$$

$$\text{for } x_1, \dot{x}_1 : \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x}_1\right) - \frac{\partial}{\partial x_1}\left(-kx_1 - 2\,(kx_2 - x_1)(-1)\right)$$
$$\Rightarrow m\ddot{x}_1 + 3kx_1 - 2kx_2 = 0$$

repeat for $i = 2, 3 \Rightarrow \mathrm{d}\mathbf{P}_2, \mathrm{d}\mathbf{P}_3$

matrix (stiffness) formulation

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F}$$

for this example

$$\mathbf{M} = m\begin{bmatrix} 1 & & \\ & 2 & \\ & & 1 \end{bmatrix}, \quad \mathbf{K} = k\begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

$\equiv$ "mass matrix" $\qquad \equiv$ "stiffness" matrix

numerical solution:

$$\dot{x}_i = v_i$$
$$\dot{v}_i = \mathrm{E-L\ equation}_i \text{ normalized by } m$$

MV1.6 Mechanical Vibrations, Multiple DOF Mode Solution

Euler- Lagrange equations

$$\mathrm{d}\mathbf{P}: \quad \mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$$

$$\text{solution}: \quad \mathbf{x}(t) = \mathbf{X}\mathrm{e}^{i\omega t}, \qquad \mathbf{X} \equiv \text{mode shape (vector)}$$

Substitute into dP, premultiply by $\mathbf{M}^{-1}$

$$[\mathbf{M}^{-1}\mathbf{K} - \omega^2\mathbf{I}]\mathbf{X} = \mathbf{0}$$

homogeneous BCs: $\det\left[\mathbf{M}^{-1}\mathbf{K} - \omega^2\mathbf{I}\right] = 0$

eigenvalues: $\omega^2 \Rightarrow \omega_i^2, \quad i = 1, 2, 3\ldots, n$

natural frequency: positive root of ω_i^2

frequency ordering: $\omega_1 \le \omega_2 \le \omega_3 \le \cdots \le \omega_n$

mode shapes: $\mathbf{X} \Rightarrow \mathbf{X}_i(\omega_i)$

since homogeneous: $\mathbf{X}_i$ is non – unique to within a constant