MVn.1 Mechanical Vibrations, Continuum Systems

Lagrangian viewpoint, mechanical continuum

for Lagrangian $L \equiv T - V = f(\dot{\mathbf{u}}, \nabla \mathbf{u}, \text{data})$, and $\mathrm{d}M = 0 = \mathrm{d}E$

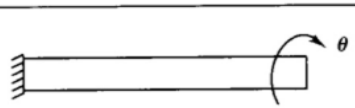
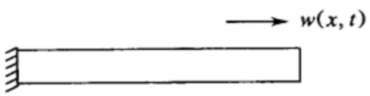
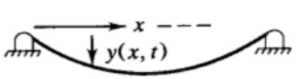
$$\mathrm{d}\mathbf{P} \Rightarrow \text{E} - \text{L equation} \;:\quad \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\mathbf{u}}}\right) - \nabla \cdot \frac{\partial L}{\partial\,(\nabla \mathbf{u})} = 0$$

Longitudinal oscillations of bar, n = 1

$$T = 1/2\,\rho\,\dot{u}^{2}, \quad V = 1/2\,\mathrm{E}\,u_{x}^{2}$$

E-L $\Rightarrow$ d**P**:

$$\mathsf{L}(u) = \ddot{u} - c^2 u_{xx} = 0, \quad c = \sqrt{\mathrm{E}/\rho}$$

Problem	Schematic	Nondimensional wave equation	Wave speed	
Torsional oscillations of circular cylinder		$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial t^2}$	$c = \sqrt{\frac{G}{\rho}}$	G = shear modulus ρ = mass density
Longitudinal oscillations of bar		$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial t^2}$	$c = \sqrt{\frac{E}{\rho}}$	E = elastic modulus ρ = mass density
Transverse vibrations of taut string		$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$	$c = \sqrt{\frac{T}{\mu}}$	T = tension μ = linear density

MVn.2 Mechanical Vibrations, *n*-D Continuum Systems

Lagrangian

$$L = T - V = f\,(\dot{\mathbf{u}}, \nabla\mathbf{u}, \text{data})$$

$$\text{E - L} \Rightarrow \text{d}\mathbf{P}: \quad \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\mathbf{u}}}\right) - \nabla \cdot \left(\frac{\partial L}{\partial \nabla \mathbf{u}}\right) = 0$$

Transverse vibrations of a plate, *n* = 2

$$\text{d}\mathbf{P}: \quad \frac{\partial^2 y}{\partial t^2} - \nabla \cdot f\,(\text{E}, \upsilon)\nabla y = 0$$

Sound propagation in free air, *n* = 3

$$\text{d}\mathbf{P}: \quad \frac{\partial^2 p}{\partial t^2} - c^2 \nabla^2 p = 0, \qquad c = \sqrt{\gamma \text{R} T}$$

Normal mode solution

$$q\,(\mathbf{x}, t) = \mathbf{Q}\,(\mathbf{x})\, e^{i\omega t}$$

eigenmodes

$$\nabla^2 \mathbf{Q}_i - \omega_i^2 \mathbf{Q}_i = \mathbf{0}$$

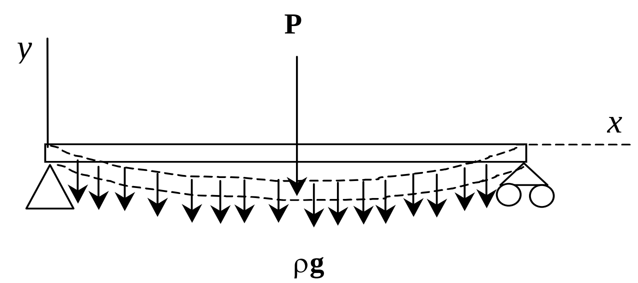
homogenous BCs

$$\omega^2 \Rightarrow \omega_i^2, \quad 1 \le i \le n\ldots, \mathbf{Q} \Rightarrow \mathbf{Q}_i$$

MVn.3 Transverse Beam, Harmonic Oscillation

Euler-Bernoulli beam, recall SM1.2

$$\mathrm{d}\mathbf{P}: \mathcal{L}(y) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\mathrm{E}I\,\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) - p(x) - \mathrm{P} = 0$$



BCs: $y(0) = 0 = y(\mathrm{L})$, pin connections
data: ρ = uniform mass density/area

Harmonic oscillation under own weight

IC: point load **P** released at $t = 0$

$$\mathrm{d}\mathbf{P}: \quad \mathcal{L}(y) \Rightarrow \rho\mathrm{A}\frac{\partial^2 y}{\partial t^2} + \mathrm{EI}\frac{\partial^4 y}{\partial x^4} = 0$$

normal mode solution: $y(x,t) = Y(x)\,e^{i\omega t}$

$$\mathrm{d}\mathbf{P}: \quad \mathcal{L}(y) \Rightarrow \mathrm{EI}\frac{\partial^4 Y}{\partial x^4} - \omega^2\rho\mathrm{A}Y = 0$$

thus: $p(x) \Rightarrow -\rho\omega^2\mathrm{A}Y(x)$, ω = frequency

MVn.4 Vibrating Beam, Normal Mode Solution

Euler-Bernoulli beam, modal solution

$$d\mathbf{P}: \; L(y) = \frac{d^4 Y}{dx^4} - \left(\frac{\rho A}{EI}\right)\omega^2 Y = 0$$

fundamental	$Y(x) \equiv C\,[B \sin(\lambda x) + D \cos(\lambda x)]$
plug into d**P**	$CB = \left(\frac{\rho A}{EI}\right) = CD\,, \quad \lambda^4 = \omega^2$
apply BCs	$Y(x=0) = 0 = Y(x=L) \Rightarrow D = 0,\, B = 1,\, \lambda = n\pi / L$
mode shape	$Y_n(x) = \left(\frac{\rho A}{EI}\right)\sin\left(\omega_n^{1/2} x\right)$
frequencies	$\omega \Rightarrow \omega_n = (n\pi / L)^2$

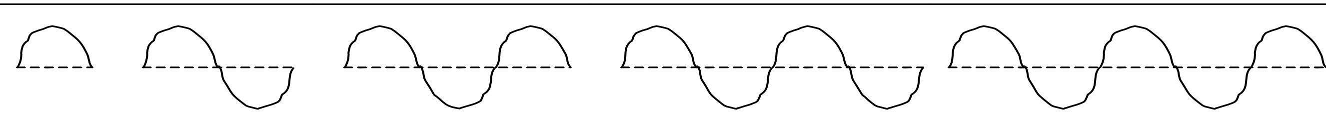
natural frequencies

n	1	2	3	4	5
ω_n	9.87	39.48	88.83	157.9	246.7

corresponding modes

n	1	2	3	4	5

y_n



MVn.5 Transverse Beam, GWS^N Formulation

E-B beam, mechanical vibration, GWS^N process

dP:

$$\mathrm{L}(y) = \rho A(x)\frac{\partial^2 y}{\partial t^2} + E\frac{\partial^2}{\partial x^2}\left(I(x)\frac{\partial^2 y}{\partial x^2}\right) + p(x) + P = 0$$

approximation

$$y(x,t) \approx y^N(x,t) \equiv \sum_{\alpha=1}^{N} \Psi_\alpha(x) Y(t)$$

Galerkin WS^N

$$\mathrm{GWS}^N \equiv \int_\Omega \Psi_\beta(x)\mathrm{L}\left(y^N\right)\mathrm{d}x \equiv 0, \quad 1 \le \beta \le M$$

$$= \int_\Omega \Psi_\beta \left[\rho A(x)\frac{\partial^2 y^N}{\partial t^2} + \frac{\partial^2}{\partial x^2} EI\frac{\partial^2 y^N}{\partial x^2} + p + P\right] dx$$

$$= \int_\Omega \left[\Psi_\beta(x)\rho A(x)\frac{\partial^2 y^N}{\partial t^2} - \frac{\mathrm{d}}{\mathrm{d}x}\Psi_\beta \frac{\mathrm{d}}{\mathrm{d}x} EI\frac{\mathrm{d}^2 y^N}{\mathrm{d}x^2} + \Psi_\beta(p + P)\right]\mathrm{d}x + \Psi_\beta \frac{\mathrm{d}}{\mathrm{d}x} EI\frac{\mathrm{d}^2 y^N}{\mathrm{d}x^2}\Bigg|_L^R$$

$$= \int_\Omega \left[\Psi_\beta(x)\rho A(x)\frac{\partial^2 y^N}{\partial t^2} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\Psi_\beta EI\frac{\mathrm{d}^2 y^N}{\mathrm{d}x^2} + \Psi_\beta(p + P)\right]\mathrm{d}x - \frac{\mathrm{d}\Psi_\beta}{\mathrm{d}x}M\Bigg|_L^R + \Psi_\beta V\Bigg|_L^R = 0$$

BCs, moment, shear

$$M \equiv EI\frac{\mathrm{d}^2 y^N}{\mathrm{d}x^2}, \quad V \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left(EI\frac{\mathrm{d}^2 y^N}{\mathrm{d}x^2}\right)$$

MVn.6 Transverse E-B Beam, $\mathrm{GWS}^N \Rightarrow \mathrm{GWS}^h$

Finite element implementation GWS^h

approximation

$$y^N(x,t) \equiv y^h(x,t) = \cup_e y_e(x,t)$$
$$y_e(x,t) \equiv \{N_k(\zeta)\}^T \{Y(t)\}_e$$

Galerkin WS^h

$$\mathrm{GWS}^h \equiv \mathrm{S}_e \{\mathrm{WS}\}_e \equiv \{0\}$$

$$\{\mathrm{WS}\}_e = \int_{\Omega_e} \{N_k\}\, \rho A(x) \{N_k\}^T \mathrm{d}x\, \{\ddot{Y}\}_e + \int_{\Omega_e} \frac{\mathrm{d}^2\{N_k\}}{\mathrm{d}x^2} \mathrm{E}I(x) \frac{\mathrm{d}^2\{N_k\}^T}{\mathrm{d}x^2} \mathrm{d}x \{Y\}_e$$
$$+ \int_{\Omega_e} \{N_k\}\, p(x)\, \mathrm{d}x - \frac{\mathrm{d}\{N_k\}}{\mathrm{d}x} \mathrm{M}\Big|_{\partial\Omega_e} + \{N_k\}\mathrm{V}\Big|_{\partial\Omega_e} + \mathrm{P}\{\delta\}$$

FE template

$$\{\mathrm{WS}\}_e = (\rho)(\mathrm{A})\{\ \ \}(1)[\mathrm{A200}]\{\ddot{Y}\}$$
$$+ (\mathrm{E})(\ \ )\{\mathrm{I}\}(-3)[\mathrm{A3022}]\{Y\}$$
$$+ (\ \ )(\ \ )\{\ \ \}(1)[\mathrm{A200}]\{P\} + \mathrm{P}\{\delta\} + \{\mathrm{BCs}\}$$

MVn.7 Euler-Bernoulli Beam, Cubic Hermite FE Basis

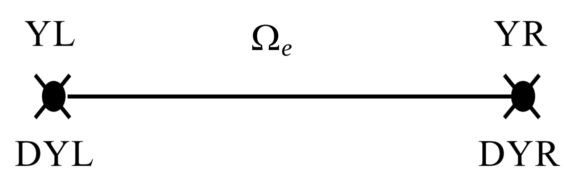
Two derivatives require $k \geq 2$ basis; let's examine

$$y_e(x) = \mathrm{a} + \mathrm{b}\left(\frac{\bar{x}}{l_e}\right) + \mathrm{c}\left(\frac{\bar{x}}{l_e}\right)^2 + \mathrm{d}\left(\frac{\bar{x}}{l_e}\right)^3 \quad , \quad \bar{x} = x - \mathrm{X_L}$$

For $\{Q\}_e$, select displacement and slope DOF

$$y_e(\bar{x} = 0) \equiv \mathrm{YL}e \quad , \quad y_e(\bar{x} = l_e) = \mathrm{YR}e$$

$$\left.\frac{\mathrm{d}y_e}{\mathrm{d}x}\right|_{\bar{x}=0} \equiv \mathrm{DYL}e \quad , \quad \left.\frac{\mathrm{d}y_e}{\mathrm{d}x}\right|_{\bar{x}=l_e} \equiv \mathrm{DYR}e$$



solving 4 × 4 system for nodal DOF yields

$$y_e(x) \equiv \{N3'\}^T \{Q\}_e \quad , \quad \{N3'\} = \begin{Bmatrix} 1 - (\zeta_2)^2 (1+2\zeta_1) \\ \zeta_2 (\zeta_1)^2 l_e \\ (\zeta_2)^2 (1+2\zeta_1) \\ -\zeta_1 (\zeta_2)^2 l_e \end{Bmatrix}, \quad \{Q\}_e = \begin{Bmatrix} \mathrm{YL} \\ \mathrm{DYL} \\ \mathrm{YR} \\ \mathrm{DYR} \end{Bmatrix}_e$$

checks out for (0,1) consistency at nodes

MVn.8 Euler-Bernoulli Beam, Hermite Basis FE Library

Using $\{N_1\}$ for $I(x)$, $p(x)$, recalling $\int_{\Omega_e} \zeta_1^p \zeta_2^q \, dx = \ell_e p!q!/(1+p+q)!$

$$[\mathrm{STIFF}]_e = \frac{E\{I\}_e^T}{l_e^3}\begin{bmatrix} \begin{Bmatrix}6\\6\end{Bmatrix} & \begin{Bmatrix}4\\2\end{Bmatrix} l_e & \begin{Bmatrix}-6\\-6\end{Bmatrix} & \begin{Bmatrix}2\\4\end{Bmatrix} l_e \\ & \begin{Bmatrix}3\\1\end{Bmatrix} l_e^2 & \begin{Bmatrix}-4\\-2\end{Bmatrix} l_e & \begin{Bmatrix}1\\1\end{Bmatrix} l_e^2 \\ & & \begin{Bmatrix}6\\6\end{Bmatrix} & \begin{Bmatrix}-2\\-4\end{Bmatrix} l_e \\ & (\mathrm{sym}) & & \begin{Bmatrix}1\\3\end{Bmatrix} l_e^2 \end{bmatrix} \equiv \frac{E\{I\}_e^T}{l_e^3}[\mathrm{A3022LH}]$$

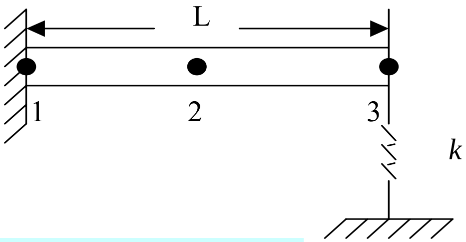
$$[\mathrm{MASS}]_e = \rho A_e \int_{\Omega_e} \{N_3'\}\{N_3'\}^T dx \equiv \frac{\rho \overline{A}_e l_e}{420}\begin{bmatrix} 156 & 22 l_e & 54 & -13 l_e \\ & 4 l_e^2 & 13 l_e & -3 l_e^2 \\ & & 156 & -22 l_e \\ (\mathrm{sym}) & & & 4 l_e^2 \end{bmatrix} = \rho \overline{A}_e l_e [\mathrm{A200H}]_e$$

$$\{\mathrm{load}\}_e \equiv \int_{\Omega_e} \{N_{3'}\} p_e(x)\, dx + \mathrm{P} = \frac{l_e}{60}\begin{bmatrix} 21 & 9 \\ 3l_e & 2l_e \\ 9 & 21 \\ -2l_e & -3l_e \end{bmatrix}\begin{Bmatrix}\mathrm{PL}\\ \mathrm{PR}\end{Bmatrix}_e + \{\mathrm{PP}\} \equiv l_e[\mathrm{A200HL}]_e\{\mathrm{P}\}_e + \{\mathrm{PP}\}$$

Moment and shear BCs

$$\{N_{3'}\} V \Big|_{\partial \Omega_e} = V \begin{Bmatrix} \delta_{e1} \\ 0 \\ \delta_{eM} \\ 0 \end{Bmatrix}, \quad \frac{d\{N_{3'}\}}{dx} M \Big|_{\partial \Omega_e} = M \begin{Bmatrix} 0 \\ \delta_{e1} \\ 0 \\ \delta_{eM} \end{Bmatrix}$$

MVn.9 Transverse Beam, GWS^h Assembly Example



data:

$\rho A = 200 \text{ kg}, \quad I = 5\times 10^{-5}\ \text{m}^4, \ L = 1\ \text{m}$

$E = 210 \times 10^9\ \text{N/m}^2, \ k_{\text{spring}} = 2\times 10^6\ \text{N / m}$

$$M = 2\ \Omega_e\,,\ \text{DOF} \Rightarrow \{Q\}_1 = \{0, 0, \text{Y2}, \text{DY2}\}^T,\ \{Q\}_2 = \{\text{Y2}, \text{DY2}, \text{Y3}, \text{DY3}\}^T$$

$$[\text{MASS}] = \text{S}_e[\text{MASS}]_e = \text{S}_e\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 156 & -22l_e \\ (\text{sym}) & & -22l_e & 4l_e^2 \end{bmatrix}\{\ddot{Q}\}_{e=1}, \begin{bmatrix} 156 & 22l_e & 54 & -13l_e \\ & 4l_e^2 & 13l_e & -3l_e^2 \\ & & 156 & -22l_e \\ (\text{sym}) & & & 4l_e^2 \end{bmatrix}\{\ddot{Q}\}_{e=2}\right)$$

$$\Rightarrow \frac{\rho A l_e}{420}\begin{bmatrix} 312 & 0 & 54 & -13l_e \\ & 8l_e^2 & 13l_e & -3l_e^2 \\ & & 156 & -22l_e \\ (\text{sym}) & & & 4l_e^2 \end{bmatrix}\begin{Bmatrix} \ddot{\text{Y}}_2 \\ \text{D}\ddot{\text{Y}}2 \\ \ddot{\text{Y}}3 \\ \text{D}\ddot{\text{Y}}3 \end{Bmatrix}$$

$$[\text{STIFF}] = \text{S}_e[\text{STIFF}]_e = \text{S}_e\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 12 & -6l_e \\ (\text{sym}) & & -6l_e & 4l_e^2 \end{bmatrix}\{Q\}_{e=1}, \begin{bmatrix} 12 & 6l_e & -12 & -6l_e \\ & 4l_e^2 & -6l_e & -2l_e^2 \\ & & 12 & -6l_e \\ (\text{sym}) & & & 4l_e^2 \end{bmatrix}\{Q\}_{e=2}\right)$$

$$\Rightarrow \left(\frac{\text{EI}}{l_e^3}\begin{bmatrix} 24 & 0 & -12 & -6l_e \\ & 8l_e^2 & -6l_e & -2l_e^2 \\ & & 12 & -6l_e \\ (\text{sym}) & & & 4l_e^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & k & 0 \\ (\text{sym}) & & & 0 \end{bmatrix}\right)\begin{Bmatrix} \text{Y2} \\ \text{DY2} \\ \text{Y3} \\ \text{DY3} \end{Bmatrix}$$

MVn.10 Transverse Beam, FE Modal Solution

GWS^h M = 2 matrix statement

$$GWS^h = [MASS]\{\ddot{Q}\} + [STIFF]\{Q\} = \{0\}$$

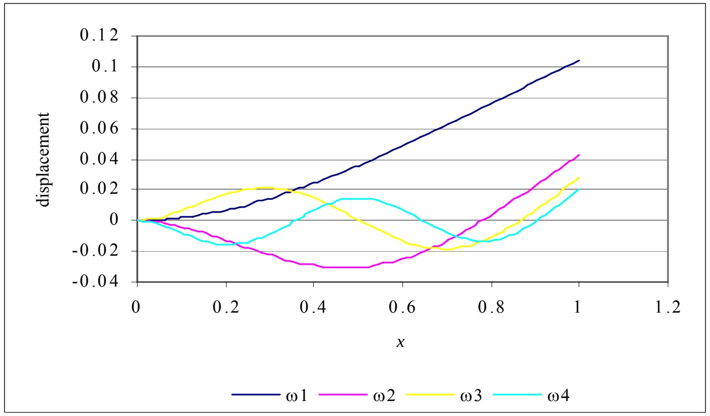
normal mode solution: $y^h(x,t) = Q(x)e^{i\omega t}$

$$GWS^h \Rightarrow \left([STIFF] - \omega^2[MASS]\right)\{Q\} = \{0\}$$

natural frequencies: $\omega \Rightarrow \omega_i^h = \sqrt{\text{eigenvalues of } [MASS]^{-1}[STIFF]}$

Matlab experiment, $4 \le M \le 64$

i	1	2	3	4	
ω_i^h	806	5.09 E3	1.72 E4	5.00 E4	M = 2
ω_i	829	5.05 E3	1.41 E4	2.73 E4	exact



displacement DOF normal modes

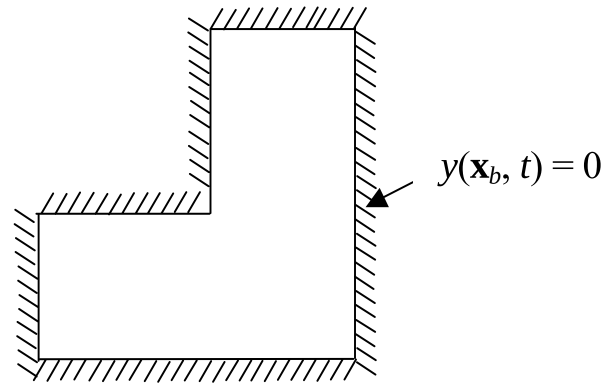
Natural Frequency

M	ω_1*1E-04	ω_2*1E-04	ω_3*1E-04	ω_4*1E-04
4	0.0838220	0.5070094	1.4249880	2.8102408
8	0.0838704	0.5067519	1.4155006	2.7769875
16	0.0838752	0.5067581	1.4148080	2.7713884
32	0.0838754	0.5067615	1.4147716	2.7710265
64	0.0838753	0.5067621	1.4147705	2.7710054

MVn.11 Transverse Vibrations, L-Shaped Membrane

Transverse vibrations of a plate

$$\mathrm{d}\mathbf{P}: \quad \frac{\partial^2 y}{\partial t^2} - \nabla \cdot f(E, \upsilon) \nabla y = 0$$



normal mode solution	$y(\mathbf{x}, t) = \mathrm{Q}(\mathbf{x})\, e^{i\omega t}$
GWS^h for eigenmodes	$\left[[\mathrm{STIFF}] - \omega^2 [\mathrm{MASS}]\right]\{Q\} = \{0\}$
homogeneous BCs	$\det([\mathrm{MASS}]^{-1} [\mathrm{STIFF}] - \omega_i^2 [\mathrm{I}]) = \{0\}$

GWS^h normal mode solutions, $\omega_i^h = 45, 71, 99$ for $i = 7, 12, 19$