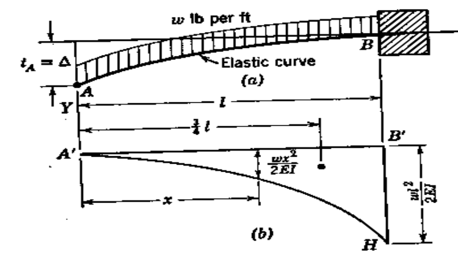
SM1.1 Structural Mechanics, Conservation Principles

Lagrangian viewpoint for elastic continuum:

$$\mathrm{d}M = 0 = \mathrm{d}E$$
$$\mathrm{d}\mathbf{P} = \sum \mathbf{F} = \mathbf{0}$$
$$\mathbf{r} \times \mathrm{d}\mathbf{P} = \sum \mathbf{M} = \mathbf{0}$$



n = 1 applications

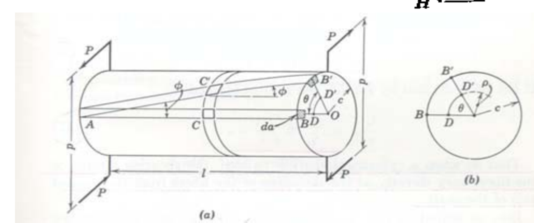
beams $: \mathrm{E}I\mathrm{d}^2 y/\mathrm{d}x^2 = M$

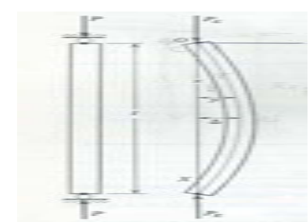
torsion $: \tau_{\max} = Tc/J$

columns $: \mathrm{E}I\mathrm{d}^2 y/\mathrm{d}x^2 = M = -Py$

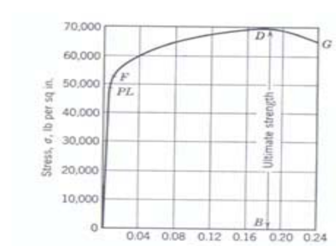
modal solution $: y(x) = \sin\left(\dfrac{n\pi x}{\ell}\right)$

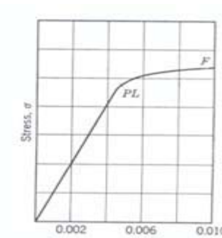
pre-buckling load $: P_b = \pi^2 \mathrm{E}I/\ell^2$





inelastic behavior:





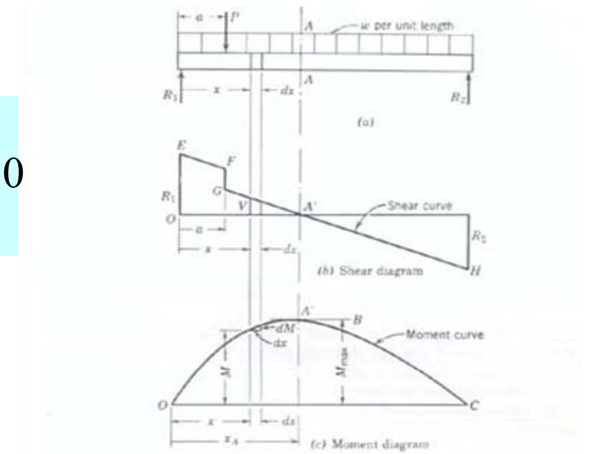
SM1.2 Euler-Bernoulli Beam

E-B beam, distributed and point loadings

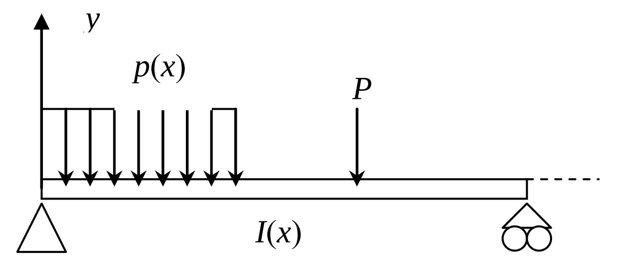
statically determinant, uniform cross-section

d**P** + Hooke:

$$\mathcal{L}(y) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\mathrm{EI}\,\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) - p\;(x) - \mathrm{P} = 0$$



data: E = Young's modulus
I = area moment of inertia
$p(x)$ = distributed load
P = point load
solution: $y\,(x)$ = vertical displacement



BCs: y or dy/dx *and*
M *or* V at x_L, x_R

for $M \equiv \mathrm{EI}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$, $V \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{EI}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)$

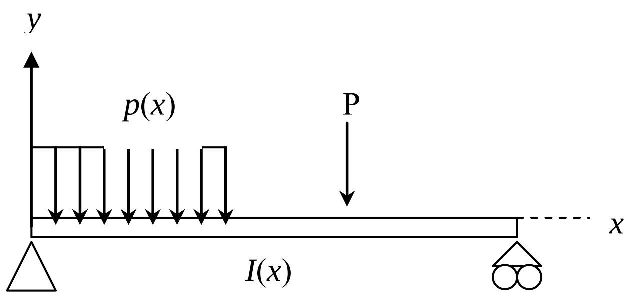
SM1.3 Euler-Bernoulli Beam, GWS^h

E-B beam, separating dP and kinematics, variable *I*

$$\mathrm{d}\mathbf{P}: M \equiv \mathrm{E}I(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$$

$$\mathbf{r} \times \mathrm{d}\mathbf{P}: \mathcal{L}(M) = -\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} - p(x) - \mathrm{P} = 0, \ \ \mathrm{BCs}: M, V$$

$$\mathrm{d}\mathbf{P} \ \ \text{alternate}: \mathcal{L}(y) = -\mathrm{E}I(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + M = 0, \ \ \mathrm{BCs}: y, \ \mathrm{d}y/\mathrm{d}x$$



GWS^h template psudo-code, for any $\{N_k(\zeta)\}$ FE basis

$$\{\mathrm{WS}\}_e = (\mathrm{const})(\mathrm{avg})_e\{\mathrm{dist}\}_e(\mathrm{det})[\mathrm{FE\ \ matrix}]\{Q \ \ \mathrm{or} \ \ \mathrm{data}\}_e$$

$$\{\mathrm{WS}(M)\}_e = (\ )(\ )\{\ \}(-1)[\mathrm{A}211]\{M\} + (\ )(\ )\{\ \}(1)[\mathrm{A}200]\{\mathrm{P}\} + \mathrm{P}\{\delta\} + \{\mathrm{BCs}\}$$

$$\{\mathrm{WS}(Y)\}_e = (\mathrm{E})(\mathrm{I})\{\ \}(-1)[\mathrm{A}211]\{Y\} + (\ )(\ )\{\ \}(1)[\mathrm{A}200]\{M\} \ + \ \{\mathrm{BCs}\}$$

SM1.4 Timoshenko Beam, GWS^h

Timoshenko beam theory, dP + Hooke with r × dP

$$\mathcal{L}(r) = -\mathrm{E}I\frac{\mathrm{d}^2 r}{\mathrm{d}x^2} - \mathrm{kG}A\left(\frac{\mathrm{d}w}{\mathrm{d}x} - r\right) = 0$$

$$\mathcal{L}(w) = -\mathrm{kG}A\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \frac{\mathrm{d}r}{\mathrm{d}x}\right) + p(x) + \mathrm{P} = 0$$

r = plane rotation
w = vertical displacement
A = cross section area
G = shear modulus, k = constant

GWS^h template pseudo-code for any $\{N_k(\zeta)\}$

$\{\mathrm{WS}\}_e$ = (const)(avg)$_e${dist}$_e$(det)[FE matrix]{Q or data}$_e$

$\{\mathrm{WS}\ (R)\}_e$ = (E)(I){ }(-1)[A211]{R} + (-kG)(A){ }(0)[A201]{W}
+ (kG)(A){ }(1)[A200]{R} +{BCs}

$\{\mathrm{WS}\ (W)\}_e$ = (kG)(A){ }(-1)[A211]{W} + (kG)(A){ }(0)[A201]{R}
+ ( )( ){ }(1)[A200]{P}+P{δ} + {BCs}

SM1.5 Timoshenko Beam GWS^h, Under-Integration

Timoshenko GWS^h implementation is "excessively stiff"

correction is to *under-integrate offending term*

$$\mathrm{GWS}^h\left(\mathcal{L}(r)\right) \Rightarrow \int_{\Omega_e} \{N\}\mathrm{kGA}r\mathrm{d}x \Rightarrow \mathrm{kGA}[\mathrm{A200U}]_e\{R\}_e$$

Produces *artificial diffusion* mechanism, e.g., for $\{N_1(\zeta)\}$

$$[\mathrm{A200U}] \equiv \frac{l_e}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = [\mathrm{A200L}]_e - [\mathrm{A2??L}]_e \qquad [\mathrm{A200L}]_e = \frac{\ell_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

hence:

$$[\mathrm{A2??L}]_e \equiv \frac{\ell_e}{12}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{\ell_e}{12}[\mathrm{A211L}] \Rightarrow \frac{\ell_e^2}{12}[\mathrm{DIFF}]_e$$

GWS^h template modification

$$\{\mathrm{WS}(R)\}_e = \{\mathrm{WS}\}_e + (\mathrm{kG}/12)(\mathrm{A})\{\quad\}(1)[\mathrm{A211}]\{R\}$$

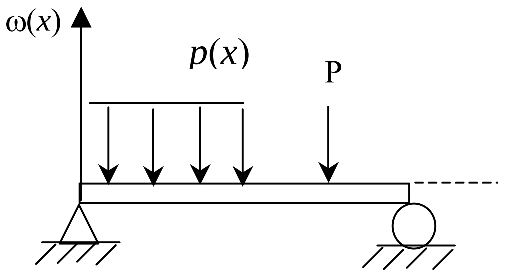
SM1.6 E-B, Timoshenko Beams, GWS^h Accuracy/Convergence

GUI creates Matlab script for either theory

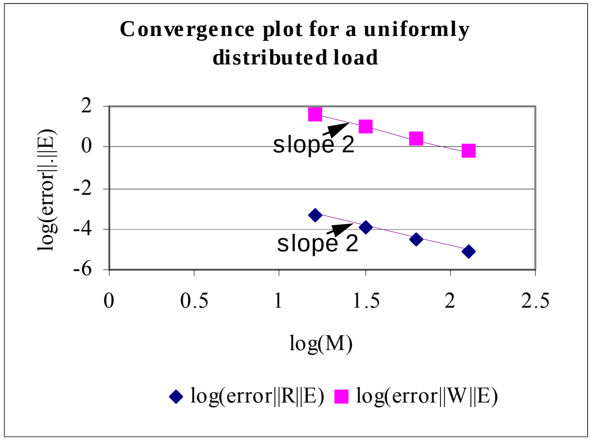
theory:

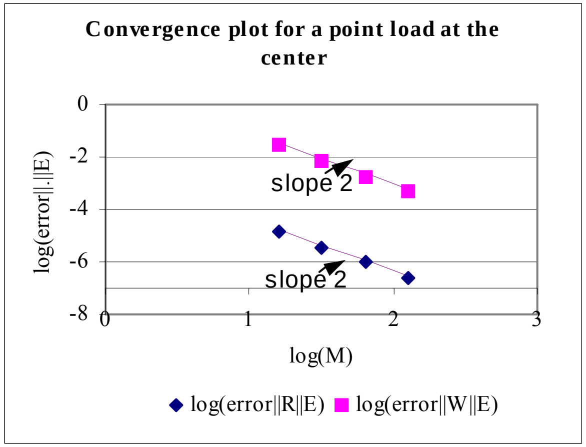
$$\left\| e^h \right\|_E \le C \ell_e^{2\gamma} \left\| \text{data} \right\|_{L2}^2$$

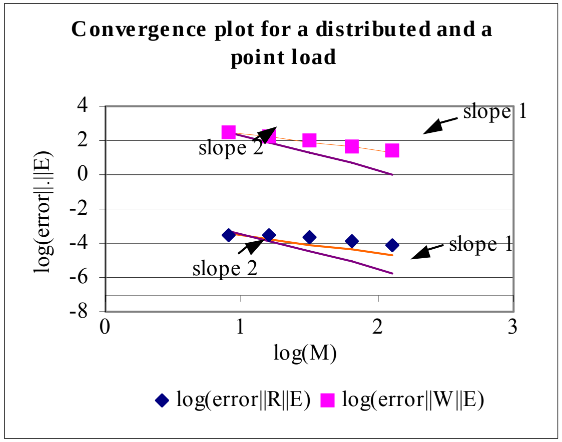
$$\gamma \equiv \min(k, r-1)$$



accuracy/convergence experiments







SM1.7 Euler-Column, Eigenmode Solution

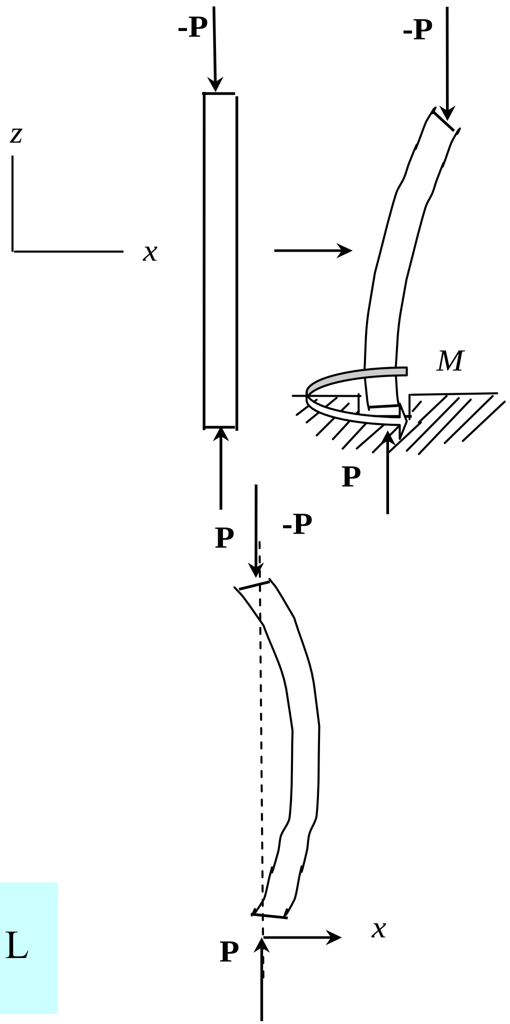
Euler column: slender vertical rod under compressive load

$$d\mathbf{P} : \mathcal{L}(x) = -\frac{d^2 x}{dz^2} + \frac{M}{EI} = 0$$

$x\,(z)$ = lateral deflection distribution

$M = |\mathbf{P}|\,x$ = bending moment

I = area moment of inertia



$$\text{BCs}: \quad \left.\begin{array}{l} x(0) = a \\ x(L) = b \end{array}\right\} \text{pin connections (if no offset } a = 0 = b)$$

$$\left.\frac{dx}{dz}\right|_{a,b} = c, \text{ clamped end } (c = 0 \text{ usually})$$

For pin connections, no offset, uniform *I*

$$d\mathbf{P} : \mathcal{L}(x) = -\frac{d^2 x}{dz^2} + \frac{|\mathbf{P}|}{EI}\,x = 0$$

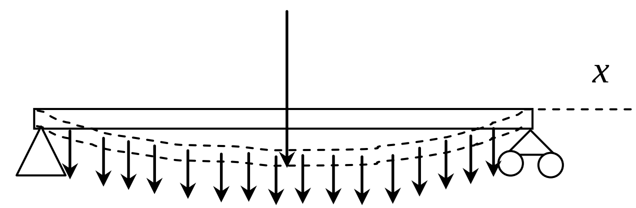
$$\text{eigenmode solution}: \; x \Rightarrow x_n(z) = \sin\left(\frac{n\pi z}{L}\right) \Rightarrow |P_n| = EI\lambda_n, \quad \lambda_n = n\pi / L$$

SM1.8 E-B Beam, Harmonic Oscillation

Euler-Bernoulli beam, harmonic oscillation

$$\text{recall}\quad \mathrm{d}\mathbf{P}: \mathcal{L}(y) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\mathrm{E}I\,\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) - p(x) - \mathrm{P} = 0$$

initial state: point load **P** released at $t = 0$



BCs: $y\,(0) = 0 = y\,(\mathrm{L})$

pin connections, restrained

data: ρ = beam mass density/length

Harmonic oscillation under own weight when P removed

then: $p\,(x) \Rightarrow \rho\omega^2 y\,(x)$, ω = frequency

D**P**: $\mathcal{L}(y) \Rightarrow$ eigenvalue problem, homogenous BCs

eigenmode solution: $y \Rightarrow y_n(x) = \sin\left(\frac{n\pi x}{\mathrm{L}}\right)$, $\omega \Rightarrow \omega_n = \left(\frac{\mathrm{E}I}{\rho}\right)^{1/2} = \lambda_n^2$