SMn.1 Structural Mechanics, *n*-Dimensions

Multi-dimensional Eulerian conservation principles

$$\mathrm{D}M = \frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho \, d\tau + \oint_{\mathrm{CS}} \rho \mathbf{V}\cdot\hat{\mathbf{n}} \, d\sigma = 0 \quad \text{identically} \; = \; \mathrm{D}E$$

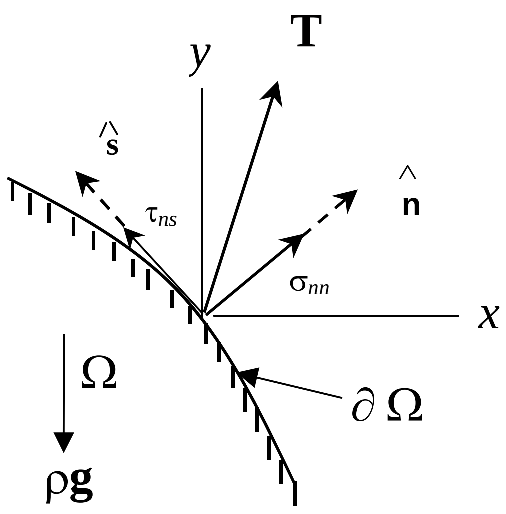
$$\mathrm{D}\mathbf{P} \Rightarrow \frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho \mathbf{V} \, d\tau + \oint_{\mathrm{CS}} \mathbf{V}\rho\mathbf{V}\cdot\hat{\mathbf{n}} \, d\sigma = \int_{\mathrm{CV}} \rho \mathbf{B} \, d\tau + \int_{\mathrm{CS}} \mathbf{T} \, d\sigma = 0$$

For arbitrary domain Ω and V = 0

$$\mathrm{D}\mathbf{P}: \; \nabla\mathbf{T} + \rho\mathbf{g} = 0$$

$$\text{stress} \;:\; \mathbf{T}_{\partial\Omega} \Rightarrow \tau_{ij}\hat{\mathbf{n}}_j \quad , \tau_{ij} \Rightarrow \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ & & \sigma_z \end{bmatrix}$$

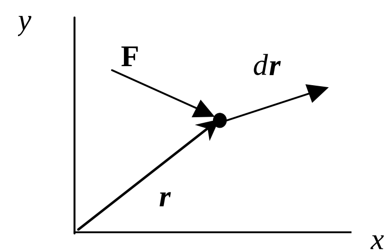
$$\mathrm{D}\mathbf{P}: \quad \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g_x = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho g_y = 0, \;\; \text{for} \;\; n = 2$$



SMn.2 Structural Mechanics, *n*-Dimensions

Energy formulation, from "virtual work"

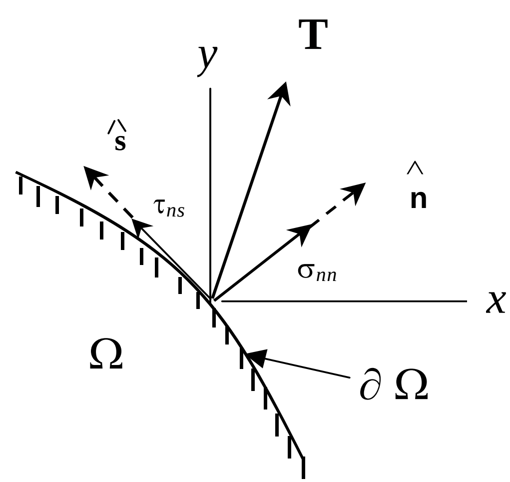
$$\mathrm{d}\mathbf{P}:\sum \mathbf{F}=0 \Rightarrow W \equiv \int_r \mathrm{d}\mathbf{P}\cdot \mathrm{d}\mathbf{r} = \int_r \mathbf{F}\cdot \mathrm{d}\mathbf{r}$$



Virtual work ⇒ variational energy principle, PS.8

in elasticity: $\mathbf{F} \Rightarrow \tau_{ij}$, $\mathrm{d}\mathbf{r} \Rightarrow \varepsilon_{ij}$

$$\mathrm{D}E: \quad \Pi \equiv \int_\Omega \mathrm{d}W = \frac{1}{2}\int_\Omega \mathrm{d}vol \int_0^\varepsilon \tau_{ij}\mathrm{d}\varepsilon_{ij} - \int_\Omega u_j B_j \mathrm{d}vol - \int_{\partial\Omega} u_j T_j \mathrm{d}surf$$



Inserting Hooke, matrix form, PS.8

$$\mathrm{D}E: \quad \Pi = \int_\Omega \left(\frac{1}{2}\{\varepsilon\}^T[\mathbf{E}]\{\varepsilon\} - \{\mathbf{u}\}^T\{\mathbf{B}\}\right)\mathrm{d}vol - \int_{\partial\Omega}\{\mathbf{u}\}^T\{\mathbf{T}\}\mathrm{d}surf$$

Π extremization ⇒ Euler-Lagrange equation, is equivalent to

$$\mathrm{DP}: \quad \nabla \mathbf{T} + \rho \mathbf{g} = 0$$

SM*n*.3 D*E* Extremum, FE implementation

Augment Π for initial stress/strain, point loads

$$\mathrm{D}E: \quad \Pi = \int_{\Omega}\left(\frac{1}{2}\{\varepsilon\}^T[\mathbf{E}]\{\varepsilon\} - \{\varepsilon\}^T[\mathbf{E}]\{\varepsilon_0\} + \{\varepsilon\}^T\{\tau_0\}\right)\mathrm{d}vol$$
$$- \int_{\Omega}\{\mathbf{u}\}^T\{\mathbf{B}\}\mathrm{d}vol - \int_{\partial\Omega}\{\mathbf{u}\}^T\{\mathbf{T}\}\mathrm{d}surf - \{\mathbf{U}\}^T\{\mathbf{P}\}$$

Approximate integral as sum over FE domains Ω_e

$$\Pi \approx \Pi^h = \sum_{\Omega_e}\Pi_e = \sum_e\left[\int_{\Omega_e}(\cdot)\,\mathrm{d}vol - \int_{\partial\Omega_e\cap\partial\Omega}(\cdot)\,\mathrm{d}surf - (\cdot)_e\right]$$

Extremize *w.r.t* non-constrained FE nodal {DOF}

$$\left(\frac{\partial\Pi^h}{\partial\{\mathrm{DOF}\}}\right) \equiv \{0\} \Rightarrow [\mathbf{K}]\{\mathbf{U}\} = \{\mathbf{R}\}$$

where: $[\mathbf{K}]$ = global “stiffness matrix”
$\{\mathbf{R}\}$ = global “load” vector
$\{\mathbf{U}\}$ = displacement vector

SM*n*.4 D*E* Extremum, FE Stiffness Matrix

D*E* extremum, plane stress, FE construction

approximation: $\mathbf{u}_e(x,y) \equiv \{N\}^T\{\mathbf{U}\}_e$

matrix form: $\{\mathbf{u}\}_e \equiv \begin{Bmatrix} u \\ v \end{Bmatrix}_e = \begin{bmatrix} \{N\}^T & \{0\} \\ \{0\} & \{N\}^T \end{bmatrix} \begin{Bmatrix} U \\ V \end{Bmatrix}_e$

kinematics: $\{\boldsymbol{\varepsilon}\} \equiv [\mathbf{D}]\{\mathbf{u}\}$, matrix derivative

$$[\mathbf{D}] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_e \equiv [\mathbf{D}]\{\mathbf{U}\} \equiv \begin{bmatrix} \partial_x & 0 \\ 0 & \partial_y \\ \partial_y & \partial_x \end{bmatrix} \begin{bmatrix} \{N\}^T & \{0\} \\ \{0\} & \{N\}^T \end{bmatrix} \begin{Bmatrix} U \\ V \end{Bmatrix}_e \equiv [\mathbf{B}]\{\mathbf{U}\}_e$$

Stiffness matrix contribution Π_e to Π^h

$$\Pi_e \equiv \int_{\Omega_e} \frac{1}{2} \{\varepsilon\}^T [\mathbf{E}] \{\varepsilon\} \, \mathrm{d}x\mathrm{d}y + \cdots$$

$$= \frac{1}{2}\{\mathbf{U}\}_e^T \int_{\Omega_e} [\mathbf{B}]^T [\mathbf{E}][\mathbf{B}] \, \mathrm{d}x\mathrm{d}y \, \{\mathbf{U}\}_e + \cdots$$

$$= \frac{1}{2}\{\mathbf{U}\}_e^T [\mathbf{k}]_e \{\mathbf{U}\}_e + \cdots, \quad [\mathbf{k}]_e \equiv \text{element stiffness matrix}$$

SM*n*.5 D*E* Extremum, Plane Elasticity, FE Matrix Statement

Extremum of Π_e for all {DOF} not BC-constrained

$$\frac{\partial \Pi_e}{\partial \{\mathbf{U}\}_e} = [\mathbf{k}]_e \{\mathbf{U}\}_e$$

Load matrix extremum

$$\{\mathbf{R}\} = \mathrm{S}_e\{\mathbf{r}\}_e, \quad \{\mathbf{r}\}_e = \int_{\Omega_e} \left([\mathbf{B}]^T [\mathbf{E}] \{\varepsilon_0\}_e - [\mathbf{B}]^T \{\sigma_0\} \right) \mathrm{d}vol$$
$$+ \int_{\Omega_e} [\{N_k\}] \left[\{N_k\}^T\right] \{\rho \mathbf{g}\} \, \mathrm{d}vol$$
$$+ \int_{\partial\Omega_e \cap \partial\Omega} [\{N_k\}] \left[\{N_k\}^T\right] \{\mathbf{T}\} \, \mathrm{d}surf \; + \{\mathbf{P}\}$$

where: $\{\mathbf{P}\}$ = point loads (must be at $\{\mathbf{U}\}_e$)
$\{\mathbf{T}\}$ = surface traction force

Extremum of DE^h is equivalent of GWS^h for DP, hence

$$\partial \Pi^h \Leftrightarrow \mathrm{GWS}^h = \mathrm{S}_e(\mathrm{WS}_e) \equiv \{0\}$$

$$\mathrm{WS}_e = [\mathrm{STIFF}]_e\{\mathrm{U}J\}_e - \{\mathbf{r}J\}_e, \quad for \;\; 1 \le J \le n = 3$$

SMn.6 GWS^h FE Template, Plane Stress/Strain

Extremum of Π^h for plane elasticity produces

$$GWS^h = S_e\{WS\}_e \equiv \{0\}$$

$$\{WS\}_e = [STIFF]_e\{UJ\}_e - \{\mathbf{r}J\}_e, \quad for \quad 1 \le J \le n = 2$$

$\{WS\}_e$ basic contributions

$$[STIFF]_e \equiv \int_{\Omega_e} [\mathbf{B}]_e^T [\mathbf{E}][\mathbf{B}]_e \, dxdy$$

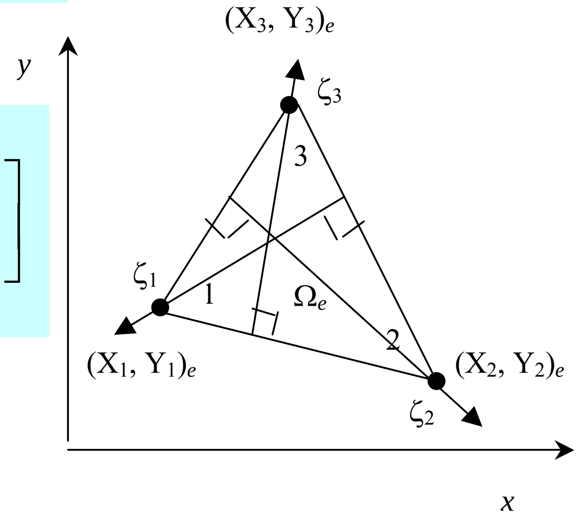
$$\{\mathbf{r}J\}_e \equiv \int_{\Omega_e} [\{N_k\}][\{N_k\}^T]\{\rho g J\}_e \, dxdy + \int_{\partial\Omega} [\{N_k\}][\{N_k\}^T]\{TJ\}_e \, d\sigma$$

GWS^h template pseudo-code, plane elasticity, any $\{N_k(\zeta,\eta)\}$

$$\begin{aligned}
\{WS\}_e &= (\text{const})(\text{avg})_e\{\text{dist}\}_e(\text{metrics})_e[\text{FE matrix}]\{Q \text{ or data}\}_e \\
&= (E)(\quad)\{\quad\}(-1)[B2KK\mathbf{E}]\{UJ\} \\
&+ (\rho)(\quad)\{\quad\}(1)[B200\mathbf{E}]\{GJ\} \\
&+ (\quad)(\quad)\{\quad\}(1)[A200\mathbf{E}]\{TJ\}
\end{aligned}$$

SMn.7 GWS^h $k = 1$ $n = 2$ FE Implementation

For $\{N_1(\zeta)\}$ on triangle Ω_e, index notation



$$[\mathbf{B}]_e \equiv \begin{bmatrix} \partial/\partial x_1 & 0 \\ 0 & \partial/\partial x_2 \\ \partial/\partial x_2 & \partial/\partial x_1 \end{bmatrix} \begin{bmatrix} \{N_1\}^T & \{0\}^T \\ \{0\}^T & \{N_1\}^T \end{bmatrix} \Rightarrow \begin{bmatrix} \{\zeta_1, \zeta_2, \zeta_3\} & \{0\} \\ \{0\} & \{\zeta_1, \zeta_2, \zeta_3\} \end{bmatrix}$$

chain rule: $\partial/\partial x_i = (\partial\zeta_\alpha/\partial x_i)_e\ \partial/\partial\zeta_\alpha$
and $\partial\{N_1\}/\partial\zeta_\alpha \Rightarrow \{\delta_\alpha\}$

$$[\mathbf{B}]_e = \begin{bmatrix} \zeta_{\alpha 1} & 0 \\ 0 & \zeta_{\alpha 2} \\ \zeta_{\alpha 2} & \zeta_{\alpha 1} \end{bmatrix}_e \frac{\partial}{\partial\zeta_\alpha} \begin{bmatrix} \{N_1\}^T & \{0\}^T \\ \{0\}^T & \{N_1\}^T \end{bmatrix} \Rightarrow \begin{bmatrix} \{\delta_\alpha\}^T & \{0\}^T \\ \{0\}^T & \{\delta_\alpha\}^T \end{bmatrix}$$

$$= \frac{1}{2A_e} \begin{bmatrix} \zeta_{11} & \zeta_{21} & \zeta_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_{12} & \zeta_{22} & \zeta_{32} \\ \zeta_{12} & \zeta_{22} & \zeta_{32} & \zeta_{11} & \zeta_{21} & \zeta_{31} \end{bmatrix}_e$$

$$\zeta_{\alpha i} = \begin{bmatrix} Y_2 - Y_3 & X_3 - X_2 \\ Y_3 - Y_1 & X_1 - X_3 \\ Y_1 - Y_2 & X_2 - X_1 \end{bmatrix}_e$$

SMn.8 GWS^h $\{N_1(\zeta)\}$ Stiffness Matrix, Plane Stress

For $\{N_1(\zeta)\}[\text{B}]_e$, and recalling SMn.6, $[\text{STIFF}]_e$, becomes

$$[\text{STIFF}]_e \equiv \int_{\Omega_e} [\mathbf{B}]_e^T [\mathbf{E}][\mathbf{B}]_e \, dxdy \equiv (\text{E})(\ )\{\ \}(\text{-}1)[\text{B}2KK\mathbf{E}]$$

$$= \frac{\text{E}\int_{\Omega_e} dxdy}{4\text{A}_e^2(1-\upsilon^2)} \begin{bmatrix} \zeta_{11} & 0 & \zeta_{12} \\ \zeta_{21} & 0 & \zeta_{22} \\ \zeta_{31} & 0 & \zeta_{32} \\ 0 & \zeta_{12} & \zeta_{11} \\ 0 & \zeta_{22} & \zeta_{21} \\ 0 & \zeta_{32} & \zeta_{31} \end{bmatrix} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & (1-\upsilon)/2 \end{bmatrix} \begin{bmatrix} \zeta_{11} & \zeta_{21} & \zeta_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_{12} & \zeta_{22} & \zeta_{32} \\ \zeta_{12} & \zeta_{22} & \zeta_{32} & \zeta_{11} & \zeta_{21} & \zeta_{31} \end{bmatrix}$$

$$= \frac{\text{Edet}_e^{-1}}{2(1-\upsilon^2)} \begin{bmatrix} \zeta_{11} & 0 & \zeta_{12} \\ \zeta_{21} & 0 & \zeta_{22} \\ \zeta_{31} & 0 & \zeta_{32} \\ 0 & \zeta_{12} & \zeta_{11} \\ 0 & \zeta_{22} & \zeta_{21} \\ 0 & \zeta_{32} & \zeta_{31} \end{bmatrix}_e \begin{bmatrix} \zeta_{11} & \zeta_{21} & \zeta_{31} & \upsilon\zeta_{12} & \upsilon\zeta_{22} & \upsilon\zeta_{32} \\ \upsilon\zeta_{11} & \upsilon\zeta_{21} & \upsilon\zeta_{31} & \zeta_{12} & \zeta_{22} & \zeta_{32} \\ (\zeta_{12} & \zeta_{22} & \zeta_{32} & \zeta_{11} & \zeta_{21} & \zeta_{31})\left(\frac{1-\upsilon}{2}\right) \end{bmatrix}_e$$

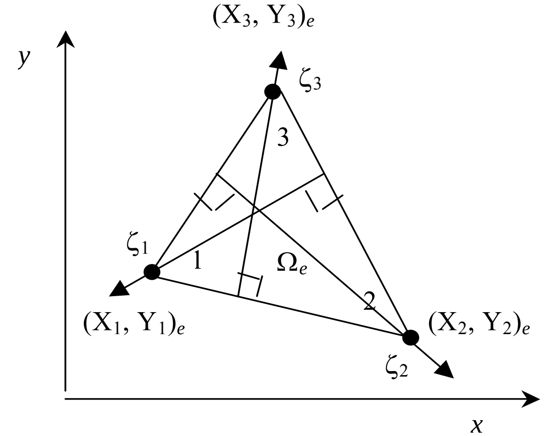
SMn.9 GWS^h $\{N_1(\zeta)\}$ Stiffness Matrix, Plane Stress

For normal-shear resolution

$$[\text{STIFF}]_e = \frac{E\det_e^{-1}}{2(1-\upsilon^2)}[\text{NORML}]_e + \frac{E\det_e^{-1}}{4(1+\upsilon)}[\text{SHEAR}]_e$$

$$[\text{NORML}]_e = \begin{bmatrix} \zeta_{11}^2 & & & & & \\ \zeta_{11}\zeta_{21} & \zeta_{21}^2 & & & (sym) & \\ \zeta_{11}\zeta_{31} & \zeta_{21}\zeta_{31} & \zeta_{31}^2 & & & \\ \upsilon\zeta_{11}\zeta_{12} & \upsilon\zeta_{12}\zeta_{21} & \upsilon\zeta_{12}\zeta_{31} & \zeta_{12}^2 & & \\ \upsilon\zeta_{11}\zeta_{22} & \upsilon\zeta_{22}\zeta_{21} & \upsilon\zeta_{22}\zeta_{31} & \zeta_{12}\zeta_{22} & \zeta_{22}^2 & \\ \upsilon\zeta_{11}\zeta_{32} & \upsilon\zeta_{32}\zeta_{21} & \upsilon\zeta_{32}\zeta_{31} & \zeta_{12}\zeta_{32} & \zeta_{22}\zeta_{32} & \zeta_{32}^2 \end{bmatrix}_e$$

$$[\text{SHEAR}]_e = \begin{bmatrix} \zeta_{12}^2 & & & & & \\ \zeta_{12}\zeta_{22} & \zeta_{22}^2 & & & (sym) & \\ \zeta_{12}\zeta_{32} & \zeta_{32}\zeta_{22} & \zeta_{32}^2 & & & \\ \upsilon\zeta_{11}\zeta_{12} & \upsilon\zeta_{11}\zeta_{22} & \upsilon\zeta_{11}\zeta_{32} & \zeta_{11}^2 & & \\ \upsilon\zeta_{21}\zeta_{12} & \upsilon\zeta_{21}\zeta_{22} & \upsilon\zeta_{21}\zeta_{32} & \zeta_{11}\zeta_{21} & \zeta_{21}^2 & \\ \upsilon\zeta_{31}\zeta_{12} & \upsilon\zeta_{31}\zeta_{22} & \upsilon\zeta_{31}\zeta_{32} & \zeta_{11}\zeta_{31} & \zeta_{21}\zeta_{31} & \zeta_{31}^2 \end{bmatrix}_e$$



$$\zeta_{\alpha i} = \begin{bmatrix} Y_2 - Y_3 & X_3 - X_2 \\ Y_3 - Y_1 & X_1 - X_3 \\ Y_1 - Y_2 & X_2 - X_1 \end{bmatrix}_e$$

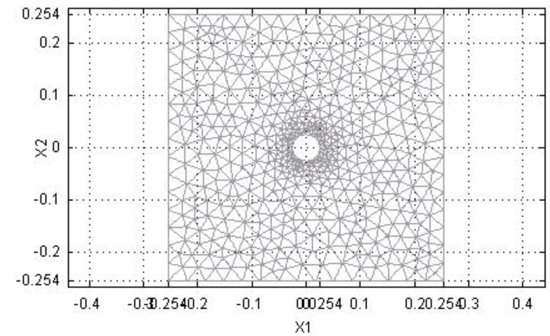
$$\{\mathbf{U}\}_e = \begin{Bmatrix} U1 \\ U2 \\ U3 \\ V1 \\ V2 \\ V3 \end{Bmatrix}_e$$

SMn.10 Plane Stress: Plate with a Hole

GWS^h for DP and/or DE extremum, plane stress, n = 2

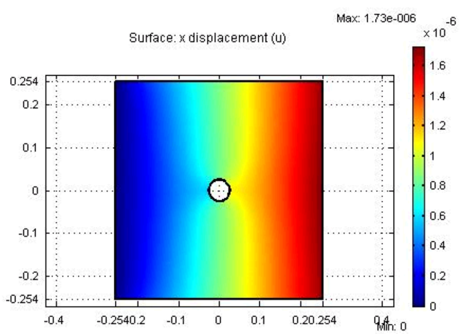
$$[\text{Matrix}\ (\mathbf{E}, \upsilon)]\begin{Bmatrix} U \\ V \end{Bmatrix} = \{\mathbf{R}\ (\varepsilon_0, \tau_0, \mathbf{T}, \mathbf{P}, \rho\mathbf{g})\}$$

meshing, Ω^h

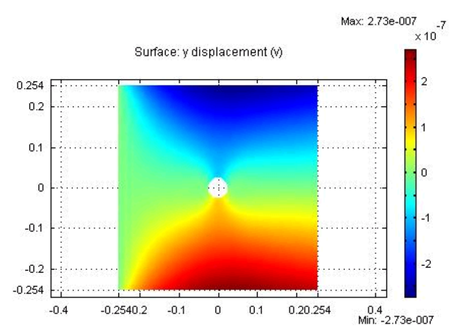


Computer lab design study

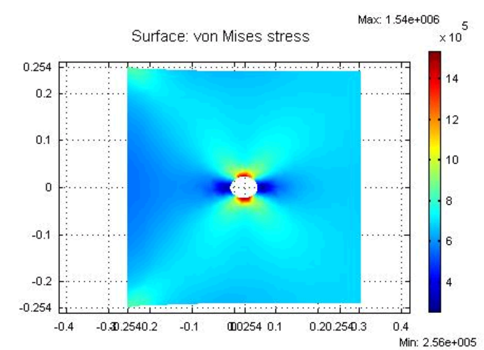
geometry: plate with hole in tension
data: L, D, **T**, BCs
solution: $u^h(x, y)$, $v^h(x, y)$
interpretation: von Mises stress concentration



x displacement, u^h



y displacement, v^h



von Mises stress