ST1.1 Unsteady Scalar Transport

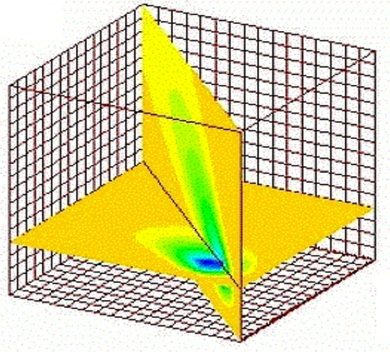
Eulerian non-D description for scalar transport

$$\mathsf{L}(q) = \frac{\partial q}{\partial t} + \mathbf{u}\cdot\nabla q - \mathrm{Pa}^{-1}\nabla\cdot(1+\mathrm{Pa}^{t})\nabla q - s = 0\,, \quad \text{on} \quad \Omega\times t$$

$$\ell(q) = \nabla q\cdot\mathbf{n} + \mathrm{Pb}(q - q_{ref}) + f_n = 0 \quad , \text{ on } \partial\Omega_r \times t$$

$$q(\mathbf{x}_b, t) = q_b(\mathbf{x}_b, t) \quad , \text{ on } \partial\Omega_b \times t$$

$$q(\mathbf{x}, t_0) = q_0(\mathbf{x}) \quad , \text{ on } \Omega \cup \partial\Omega \times t_0$$



Definitions for q(x,*t*), Pa, Pb, Pa^{t} depend on application

Transport	q	Pa	Pb	Pa^{t}
heat	Θ	RePr	Nu	$\mathrm{Re}^{t}/\mathrm{Pr}^{t}$
mass	Y	ReSc	Pa^{-1}	$\mathrm{Re}^{t}\mathrm{Sc}^{t}$
species	Y_α	ReSc_α	Pa^{-1}	$\mathrm{Re}^{t}\mathrm{Sc}^{t}_\alpha$

$$\mathrm{Re}^{t} \equiv \left(\frac{\nu^{t}}{\nu}\right)_{\mathrm{dim}}$$

ST1.2 Unsteady Energy Transport, *n* = 1

Eulerian description, energy transport, *n* = 1

DE:

$$\mathrm{L}(T) = \rho c_p \frac{\partial T}{\partial t} + \rho c_p u \frac{\partial T}{\partial x} - \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) - s = 0 \quad \text{, on } \Omega \times t \subset \Re^n \times \Re^1$$

$$\ell(T) = k \frac{\mathrm{d}T}{\mathrm{d}x} + h(T - T_r) + f_n = 0 \quad \text{, on } \partial\Omega_r \times t$$

$$T(x_b, t) = T_b(x_b, t) \quad \text{, on } \partial\Omega_b \times t$$

$$T(x, t_0) = T_0(x) \quad \text{, on } \Omega \cup \partial\Omega \times t_0$$

Time averaged, non-D by reference state (L, U, ρ, c_p, *k*)

$$\mathrm{L}(\Theta) = \frac{\partial \Theta}{\partial t} + \bar{u} \frac{\partial \Theta}{\partial x} - \frac{1}{\mathrm{Pe}} \frac{\partial}{\partial x}\left((1 + \mathrm{Re}^t) \frac{\partial \Theta}{\partial x}\right) - s_\Theta = 0$$

$$\ell(\Theta) = \frac{\mathrm{d}\Theta}{\mathrm{d}x} + \mathrm{Nu}(\Theta - \Theta_r) + f_{n\Theta} = 0$$

potential temperature: $\Theta \equiv (T - T_{min}) / (T_{max} - T_{min})$

Peclet Number: $\mathrm{Pe} = \rho c_p UL/k = \mathrm{PrRe}$

Nusselt Number: $\mathrm{Nu} = hL / k$

Re^t (turbulence): $\mathrm{Re}^t = (\nu^t/\nu)_{\mathrm{dim}}$

velocity: $\bar{u} = u(x, t) / \mathrm{U}$

ST1.3 Unsteady Energy Transport, *n* = 1 GWS^h

Approximation now contains time dependence

$$\Theta^N(x,t) \equiv \sum_{\alpha}^{N} \Psi_\alpha(x) Q_\alpha(t) \Rightarrow \Theta^h(x,t) \equiv \cup_e \Theta_e(x,t)$$

$$\text{retaining } \{N_k(\zeta,\eta)\} \text{ for all } n: \quad \Theta_e(x,t) \equiv \{N_k(\zeta)\}^T \{Q(t)\}_e$$

Weak statement GWS^N $\Rightarrow$ GWS^h

$$\mathrm{GWS}^N \equiv \int_\Omega \Psi_\beta \mathrm{L}(\Theta^N)\mathrm{d}x = \int_\Omega \Psi_\beta \left(\frac{\partial \Theta^N}{\partial t} + u\frac{\partial \Theta^N}{\partial x} - \frac{1}{\mathrm{Pe}}\frac{\partial}{\partial x}\left(\left(1+\mathrm{Re}^t\right)\frac{\partial \Theta^N}{\partial x} \right) - s_\Theta \right) \mathrm{d}x$$

$$\mathrm{GWS}^h = \mathrm{S}_e\left([\mathrm{MASS}]_e \frac{\mathrm{d}\{Q\}_e}{\mathrm{d}t} + \left([\mathrm{UVEL}]_e + [\mathrm{DIFF}]_e + [\mathrm{HBC}]_e \right)\{Q\}_e - \{\mathrm{b}\}_e \right)$$

$$= [\mathrm{MASS}]\{Q\}' + \{\mathrm{RES}\} = \{0\}$$

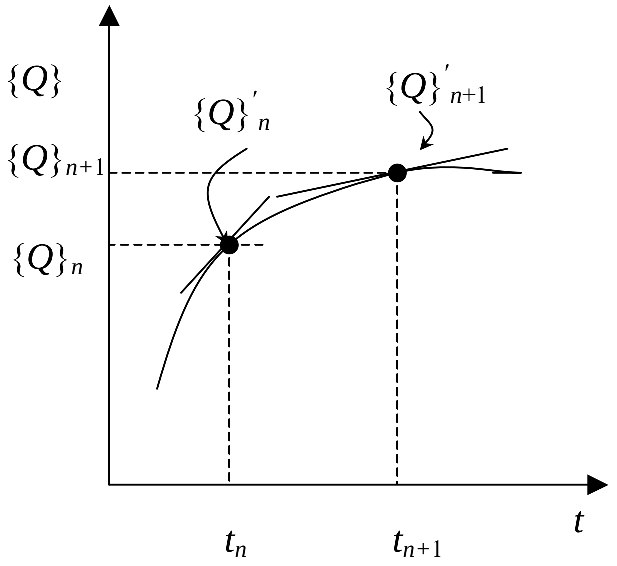
$$\{\mathrm{RES}\} = \mathrm{S}_e\left[\left([\mathrm{UVEL}]_e + [\mathrm{DIFF}]_e + [\mathrm{HBC}]_e \right)\{Q\}_e - \{\mathrm{b(data)}\}_e \right]$$

ST1.4 Unsteady GWS^h ODE System Utilization

GWS^h has produced an ODE system

$$\mathrm{GWS}^h \Rightarrow \{Q\}' = -[\mathrm{MASS}]^{-1}\{\mathrm{RES}\}$$

Use of this ODE in time Taylor series



$$\{Q(t_{n+1})\} = \begin{cases} \{Q\}_n + \Delta t\{Q\}'_n + O(\Delta t^2) \\ \{Q\}_n + \Delta t\{Q\}'_{n+1} + O(\Delta t^2) \\ \{Q\}_n + \Delta t/2\{Q'_{n+1} + Q'_n\} + O(\Delta t^3) \end{cases}$$

Linear combination ⇒ one-step "Euler family"

$$\{Q\}_{n+1} = \{Q\}_n + \Delta t\left(\theta\{Q\}'_{n+1} + (1-\theta)\{Q\}'_n\right) + O\left(\Delta t^{f(\theta)}\right)$$

implicitness parameter: $0 \le \theta \le 1.0$

ST1.5 GWS^h + θTS ⇒ Algebraic Form

Substituting GWS^h into θTS

$$\{Q\}_{n+1} = \{Q\}_n - \Delta t\,[\mathrm{MASS}]^{-1}\left(\theta\,\{\mathrm{RES}\}_{n+1} + (1-\theta)\{\mathrm{RES}\}_n\right) + \mathrm{TE}$$

To produce computable form

- multiply through by [MASS] (no inverse!)
- clear $\{\mathrm{RES}\}_{n+1}$ to left-hand side
- define change variable $\{\Delta Q\} \equiv \{Q\}_{n+1} - \{Q\}_n$

Matrix statement for computing

$$\left([\mathrm{MASS}] + \theta\Delta t[\mathrm{UVEL} + \mathrm{DIFF} + \mathrm{HBC}]\right)\{\Delta Q\} = -\Delta t\{\mathrm{RES}\}_n$$

$$\{\mathrm{RES}\}_n = [\mathrm{UVEL} + \mathrm{DIFF} + \mathrm{HBC}]\,\{Q\}_n - \{\mathrm{b}(data)\}_{n+\theta}$$

ST1.6 GWS^h + θTS Template, n = 1

Summary: GWS^h + θTS for energy transport

DE:

$$\mathrm{L}(\Theta) = \Theta_t + \bar{u}\Theta_x - \mathrm{Pe}^{-1}\left((1+\mathrm{Re}^t)\Theta_x\right)_x - s_\Theta = 0$$

$$\ell(\Theta) = \Theta_x + \mathrm{Nu}(\Theta - \Theta_r) + f_{n\Theta} = 0$$

$$\mathrm{GWS}^h + \theta\mathrm{TS} \Rightarrow \mathrm{S}_e\{\mathrm{WS}\}_e = \{0\} \Rightarrow [\mathrm{JAC}]\{\Delta Q\} = -\Delta t\{\mathrm{RES}\}_n$$

$$[\mathrm{JAC}]_e = [\mathrm{MASS}]_e + \theta\Delta t[\mathrm{UVEL} + \mathrm{DIFF} + \mathrm{HBC}]_e$$

$$\{\mathrm{RES}\}_e = \{\mathrm{UVEL} + \mathrm{DIFF} + \mathrm{HBC}]_e\{Q\}_n - \{\mathrm{b}(s_\Theta, \Theta_r, \mathrm{f}_n)\}_{n+\theta}$$

Template pseudo-code:

$$\{\mathrm{WS}\}_e = (\mathrm{const})(\mathrm{avg})_e\{\mathrm{dist}\}_e^T(\mathrm{metric})_e[\mathrm{Matrix}]\{Q \text{ or data}\}_e$$

$$[\mathrm{JAC}]_e = (\ )(\ )\{\ \}(1)[\mathrm{A200}][\ ] + (\theta\Delta t)(\ )\{\mathrm{U}\}(0)[\mathrm{A3001}][\ ]$$
$$+ (\theta\Delta t, \mathrm{Pe}^{-1})(\ )\{1+\mathrm{Ret}\}(-1)[\mathrm{A3011}][\ ]$$
$$+ (\theta\Delta t, \mathrm{Pe}^{-1}, \mathrm{Nu})(\ )\{\ \}(\ )[\mathrm{ONE}][\ ]$$

$$\Delta t\{\mathrm{RES}\}_e = (\Delta t)(\ )\{\mathrm{U}\}(0)[\mathrm{A3001}]\{\mathrm{QN}\} + (\Delta t, \mathrm{Pe}^{-1})(\ )\{1+\mathrm{Ret}\}(-1)[\mathrm{A3011}]\{\mathrm{QN}\}$$
$$+ (\Delta t, \mathrm{Pe}^{-1}, \mathrm{Nu})(\ )\{\ \}[\mathrm{ONE}]\{\mathrm{QN}\} + (-\Delta t)(\ )\{\ \}(1)[\mathrm{A200}]\{\mathrm{SRC}\}$$
$$+ (-\Delta t, \mathrm{Pe}^{-1})(\ )\{\ \}(\ )[\mathrm{ONE}]\{\mathrm{QR} - \mathrm{FN}\}$$

ST1.7 Error Estimates, n = 1 Unsteady GWS^h + θTS

For *any* solution $q^h(x, t)$ for unsteady L$(q$, Pe, Re$^t)$

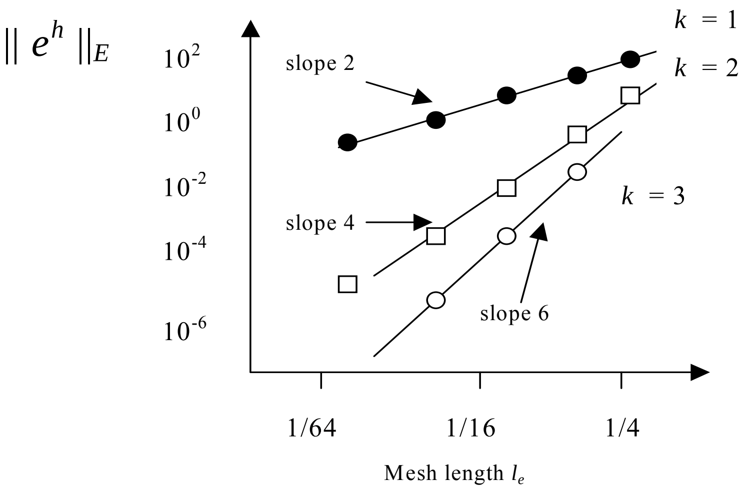
$$e^h(x, t) \equiv q(x, t) - q^h(x, t)$$

Asymptotic error estimates are *f* (Pe, $\{N_k\}$, θ, data)

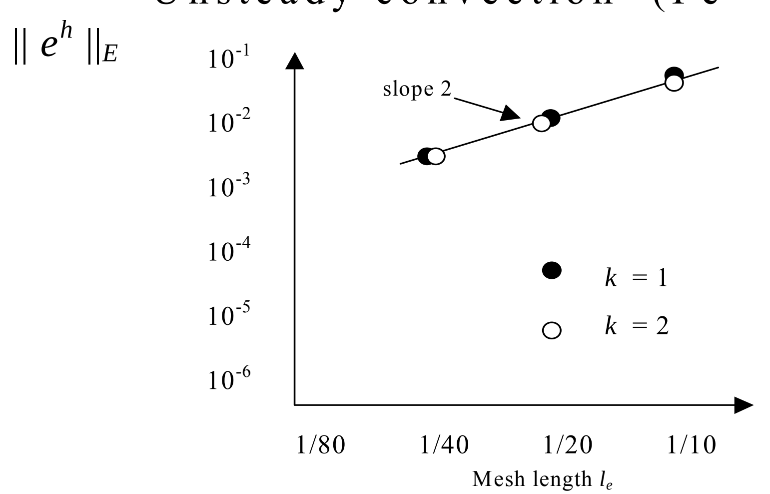
$$\mathrm{Pe}^{-1} > 0 \quad : \left\| e^h(t) \right\|_E \le C \ell_e^{2\gamma} \left\| \mathrm{data} \right\|_{L2}^2 + C_t \Delta t^{f(\theta)} \left\| q_0 \right\|_E \ , \gamma = \min(k, r-1)$$

$$\mathrm{Pe}^{-1} = 0 \quad : \left\| e^h(t) \right\|_E \le C \ell_e^2 \int_{t_0}^{t} \left\| q(x, \tau) \right\|_{H^{k+1}}^2 d\tau + C_t \Delta t^{f(\theta)} \left\| q_0 \right\|_E$$

Unsteady conduction ($\mathrm{Pe}^{-1} > 0$)



Unsteady convection ($\mathrm{Pe}^{-1} = 0$)



ST1.8 Peclet Problem, Dispersion Error

Problem statement

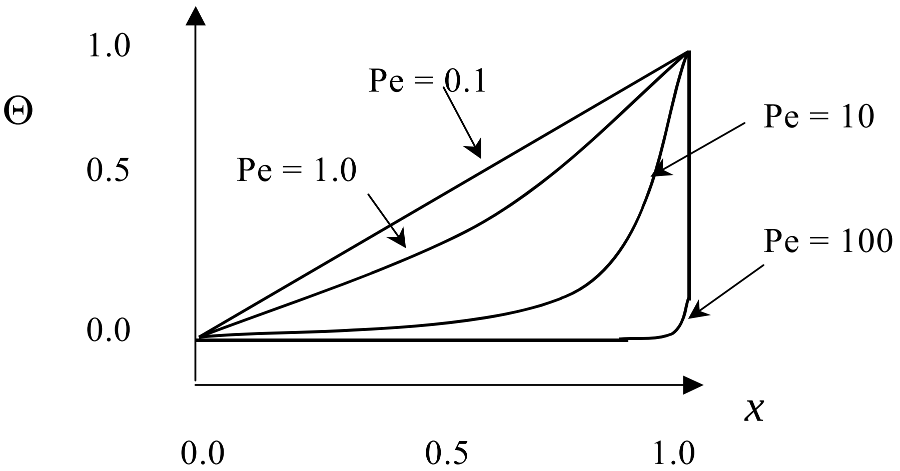
DE: $\mathrm{L}(\Theta) = \dfrac{d\Theta}{dx} - \dfrac{1}{\mathrm{Pe}} \dfrac{d^2\Theta}{dx^2} = 0$

BCs: $\Theta(0) = 0, \quad \Theta(1) = 1$

Analytical solution

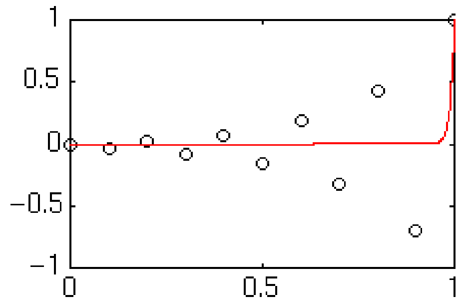
$$\Theta(x) = \frac{1 - \exp(x\,\mathrm{Pe})}{1 - \exp(\mathrm{Pe})}$$

Solution graph

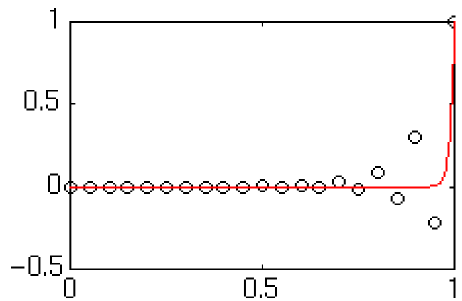


GWS^h solutions, Pe = 10^2

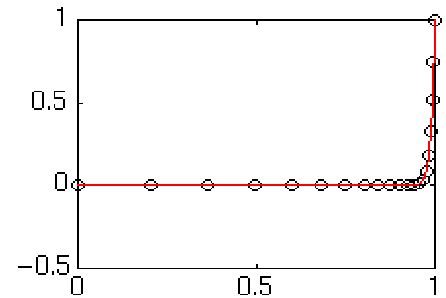
uniform Ω^h, M = 10, k = 1



uniform Ω^h, M = 10, k = 2



non-uniform Ω^h, M = 20, k = 1, 2



ST1.9 Traveling Wave, Dispersion Error

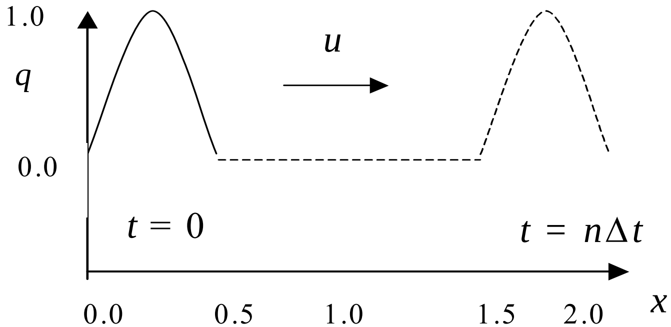
Problem statement

DM: $\mathrm{L}(q) = \frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} = 0$

BC: $q(x = 0, t) = 0$

IC: $q(x, t = t_0) = q_0(x)$

Solution graph



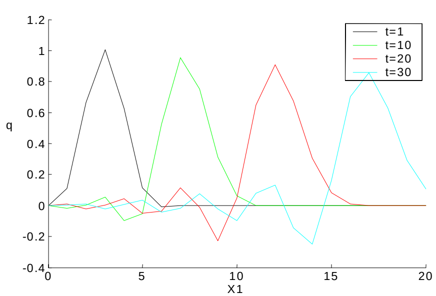
Courant No : $C \equiv u\Delta t / \Delta x$

Analytical (characteristic) solution

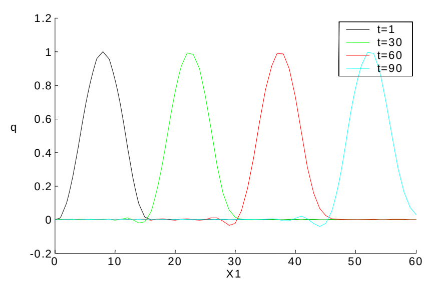
$$q(x, t) = q_0 \exp i(x - ut)$$

GWS^h + θTS solutions, k = 1 basis

$M = 20,\ C = 0.5 = \theta$



$M = 60,\ C = 0.5 = \theta$



$M = 60,\ C = 1.0 = \theta$