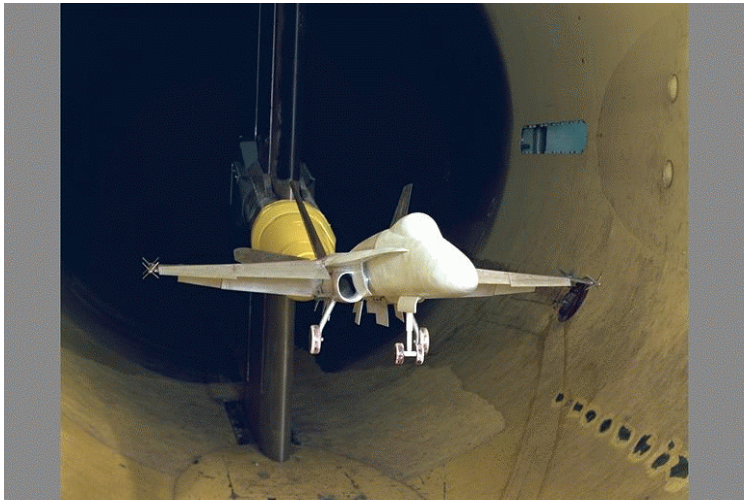
SU.1(FE.1) Engineering Simulation

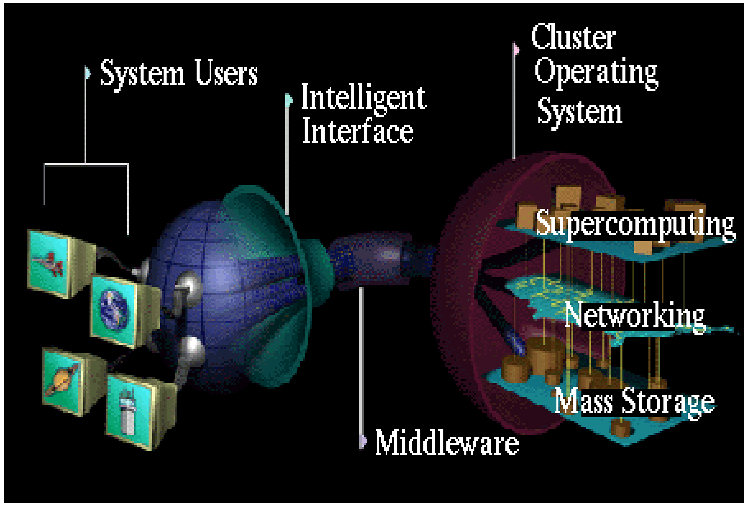
Physical Laboratory:

- ***model*** the geometry
  - similitude
  - cost
- ***measure*** the data
  - interpolation (errors)
  - ***interpretation***



Computational Laboratory:

- ***model*** the mathematics
  - conservation, BCs
- ***model*** the physics
  - complexity, cost
- ***compute*** the data
  - approximation error
  - physics model error
  - ***interpretation***



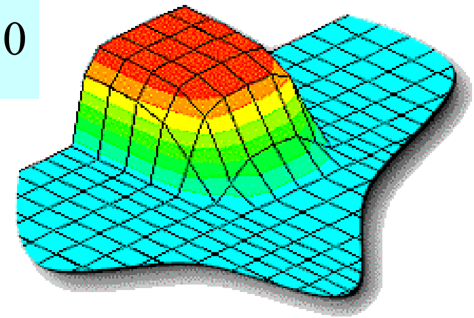
SU.2(FE.5) Summary, Finite Element Analysis

For arbitrary geometries and non-linearity

problem statement: $\mathrm{L}(q) = 0 \quad on \quad \Omega \subset \Re^n \quad + \quad \mathrm{BCs}$

approximation: $q(\mathbf{x}) \approx q^N(\mathbf{x}) \equiv \sum_{\alpha}^{N} \Psi_\alpha(\mathbf{x})\, Q_\alpha$

error minimization: $\mathrm{GWS}^N = \int_{\Omega_e} \Psi_\alpha(\mathbf{x})\, \mathrm{L}(q^N) d\tau \equiv 0$



FE discretization: $\Omega \approx \Omega^h = \cup_e\, \Omega_e$

$q^N \equiv q^h = \cup_e \{N(\mathbf{x})\}^T \{Q\}_e$

FE GWS^h: $[\ \mathrm{Matrix}\ ]\{Q\} = \{b\}$

error quantization: refined Ω^h solutions

SU.3(PS.12) Computational Simulation

Engineering design problems: PDEs + physics + BCs

unknown called *state variable* ≡ $q\ (\mathbf{x},\ t)$

solution is distribution of q on $(\mathbf{x},\ t > t_0\ )$

analytically *intractible*!

Computer simulation ⇒ seek an *approximate* solution

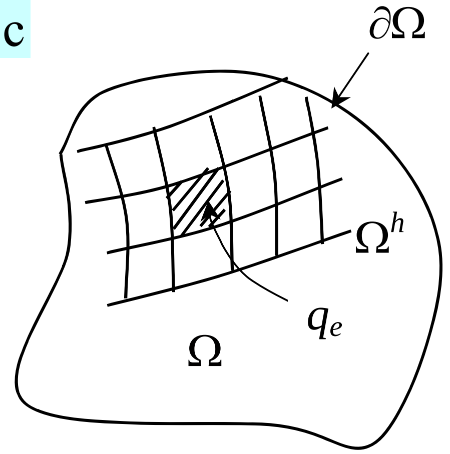
$$q(\mathbf{x},t) \approx q^N(\mathbf{x},t)$$

finite difference - historical, archaic

finite element analysis

$$q^N(\mathbf{x},t) \equiv q^h(\mathbf{x},t) = \cup_e q_e(\mathbf{x},t)$$

optimal
encompassing
real world problems



SU.4(HC.14) Error Estimation, Energy Norm

Improved error estimate uses entire solution via a "norm"

$$\textit{energy norm} \equiv \|T\|_E^h \equiv \frac{1}{2}\int_\Omega k \frac{dT^h}{dx}\frac{dT^h}{dx}d\tau \Rightarrow \frac{1}{2}\sum_e^M \{Q\}_e^T [\mathrm{DIFF}]_e \{Q\}_e$$

Uniform mesh refinement study

$$\|T^h\|_E + \|e^h\|_E = \|T\|_E = \|T^{h/2}\|_E + \|e^{h/2}\|_E = \ldots$$

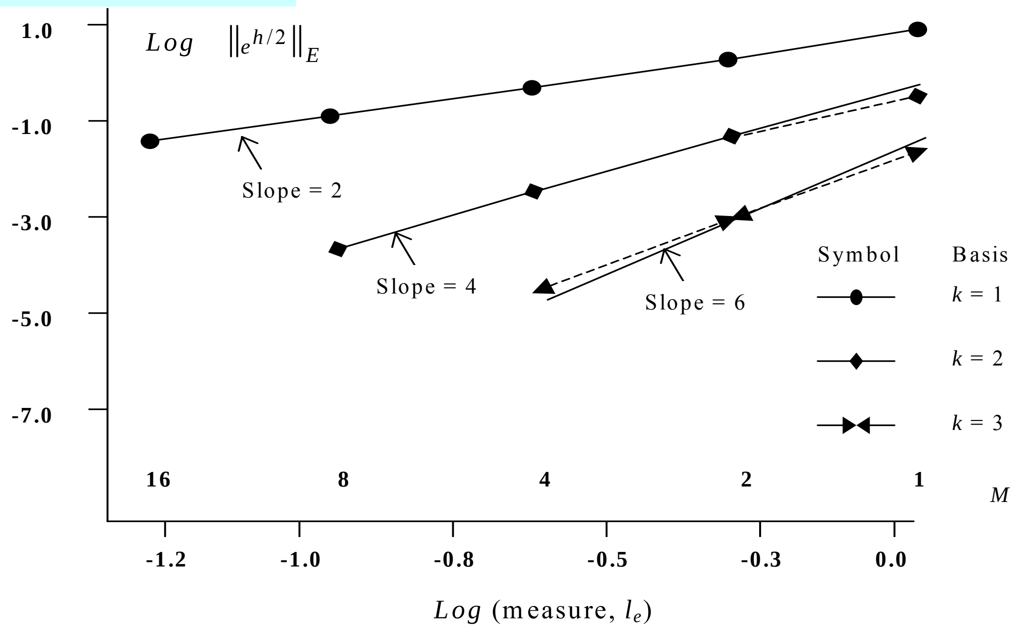
$$\textit{asymptotic convergence}: \|e^h\|_E \le C_k \ell_e^{2k} \|\mathrm{data}\|_{L2}^2$$

error estimator

$$\|e^{h/2}\|_E = \frac{\Delta\|T^{h/2}\|_E}{2^{2k}-1}$$

confirmation of theory

$$\textit{slope} = \frac{\log \|e^{h/M}\|_E / \|e^{h/2M}\|_E}{\log 2}$$



SU.5(HT1.12) D*E* GWS^h Summary, *n* = 1

Given D*E* + BC problem statement on *n* = 1

$$\mathrm{L}(q) = 0 \; on \; \Omega \subset \Re^1, \qquad \ell(q) = 0 \quad on \; \partial\Omega$$

FE weak statement recipe

approximation: $T(x) \approx T^N(x) \equiv T^h(x) = \cup_e T_e(x)$

FE basis: $T_e(x) = \{N_k(\zeta)\}^T \{Q\}_e$

error extremization: $\mathrm{GWS}^N = \int_\Omega \Psi_\beta(x) \mathrm{L}\left(T^N\right) dx \equiv \{0\} \Rightarrow \mathrm{GWS}^h = \mathrm{S}_e \{\mathrm{WS}\}_e$

matrix statement: $\{\mathrm{WS}\}_e = \left([\mathrm{DIFF}]_e + [\mathrm{BCs}]_e\right)\{Q\}_e - \{\mathrm{b}(\mathrm{data})\}_e$

error estimation: $\left\|e^h\right\|_E \leq \mathrm{C}\,\ell_e^{2\gamma} \left\|\mathrm{data}\right\|_{\mathrm{L}2}^2, \quad \gamma \equiv \min(\,k + 1 - m,\, r - m)$

FE *template* pseudo-code

$$\{\mathrm{WS}\}_e = (\mathrm{const})\,(\mathrm{avg})_e \{\mathrm{dist}\}_e^T (\mathrm{metric})_e [\mathrm{FE\ matrix}] \{Q \; or \; \mathrm{data}\}_e$$

SU.6(HTn.8) Summary, n-D GWS^h Essence for DE

FE discrete implementation GWS^h for steady DE

"recipe" $\Rightarrow$ analytical transformation of PDE plus BCs to algebraic (computable) form

analogous $\Rightarrow$ transformation methods for linear PDEs

solution^h $\Rightarrow$ parametric function of Re, Gr, Pr, Nu and *data*

error^h $\Rightarrow$ controllable via Ω^h and $\{N_k(\zeta,\eta)\}$ selections

$$\left\| e^h \right\|_E \le C h^{2\gamma} \left\| \text{data} \right\|_{L2}^2$$

$$\gamma \equiv \min(k+1-m,\ r-m)$$

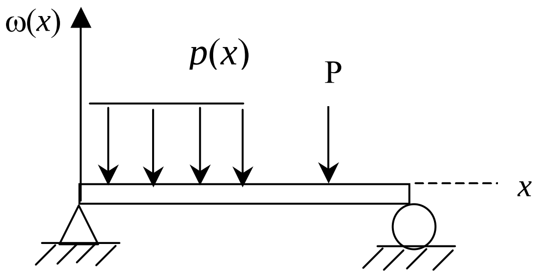
SU.7(SM1.6) E-B, T Beams, GWS^h Accuracy/Convergence

GUI creates Matlab script for either theory

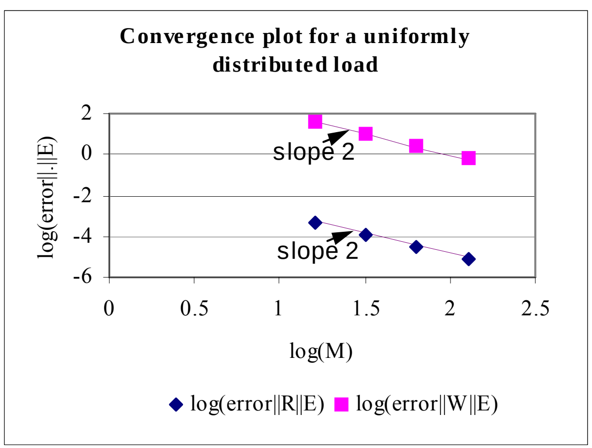
theory:

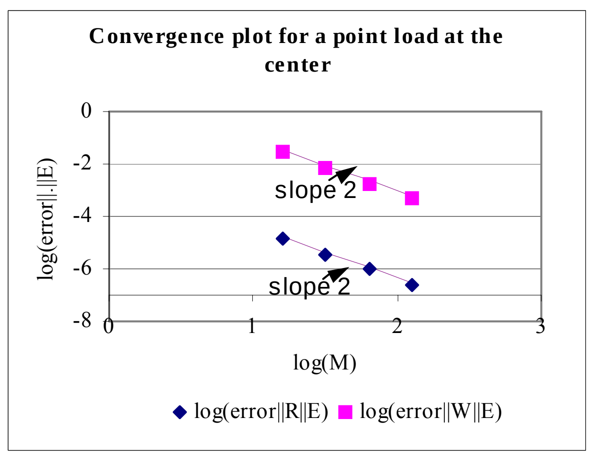
$$\left\| e^h \right\|_E \le C \ell_e^{2\gamma} \left\| \text{data} \right\|_{L2}^2$$

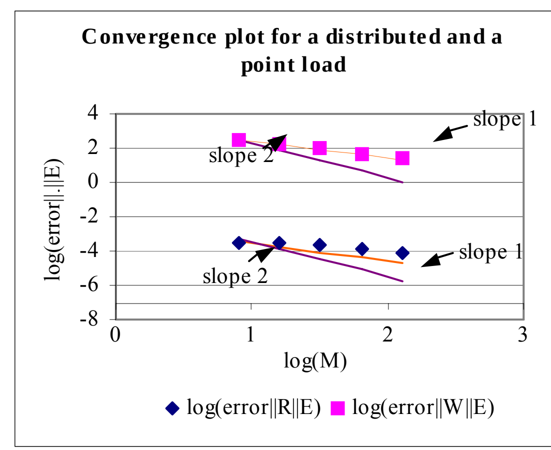
$$\gamma \equiv \min(k, r-1)$$



accuracy/convergence experiments





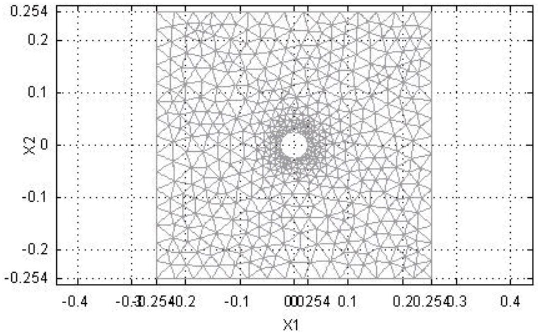


SU.8(SM*n*.10) Plane Stress: Plate with a Hole

GWS^h for DP and/or D*E* extremum, plane stress, *n* = 2

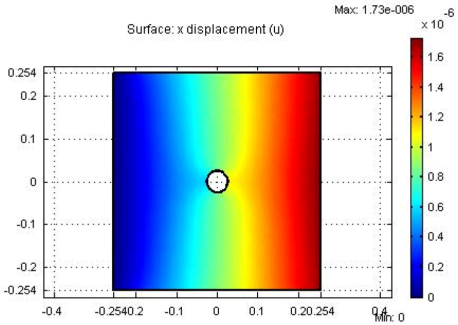
$$[\text{Matrix }(\mathbf{E}, \upsilon)]\begin{Bmatrix} U \\ V \end{Bmatrix} = \{\mathbf{R}\,(\varepsilon_0, \tau_0, \mathbf{T}, \mathbf{P}, \rho\mathbf{g})\}$$

meshing, Ω^h

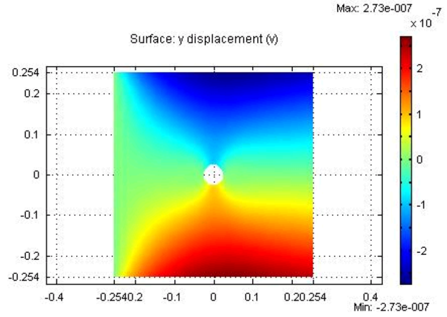


Computer lab design study

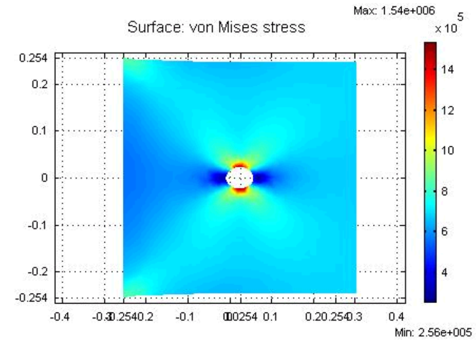
geometry: plate with hole in tension
data: L, D, **T**, BCs
solution: $u^h(x, y)$, $v^h(x, y)$
interpretation: von Mises stress concentration



x displacement, u^h



y displacement, v^h

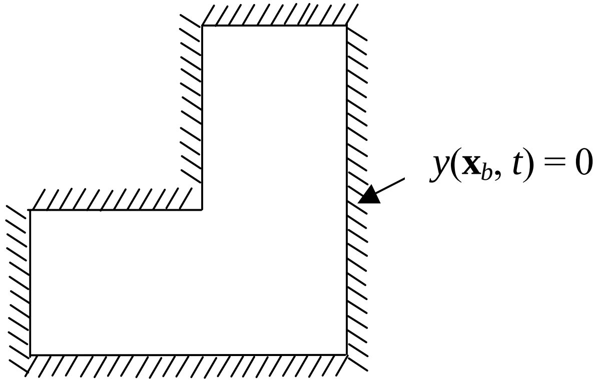


von Mises stress

SU.9(MVn.11) Mechanical Vibrations Normal Modes

Transverse vibrations of a plate

$$\mathrm{d}\mathbf{P}: \quad \frac{\partial^2 y}{\partial t^2} - \nabla \cdot f(E, \upsilon) \nabla y = 0$$

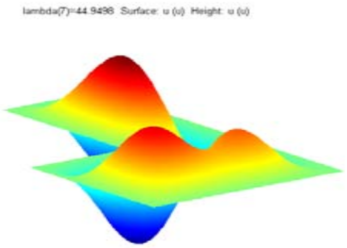


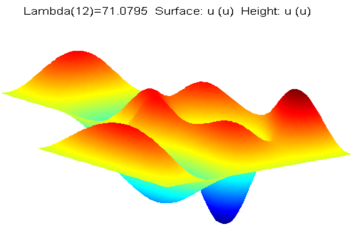
normal mode solution: $y(\mathbf{x}, t) = \mathrm{Q}(\mathbf{x})\, e^{i\omega t}$

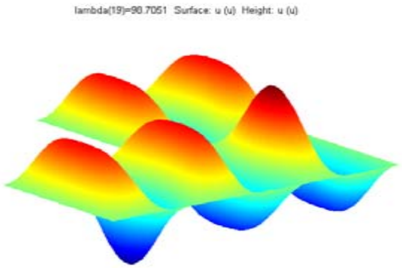
GWS^h for eigenmodes: $\left[[\mathrm{STIFF}] - \omega^2 [\mathrm{MASS}]\right]\{Q\} = \{0\}$

homogeneous BCs: $\det([\mathrm{MASS}]^{-1} [\mathrm{STIFF}] - \omega_i^2 [\mathrm{I}]) = \{0\}$

GWS^h normal mode solutions, $\omega_i^h = 45, 71, 99$ for $i = 7, 12, 19$



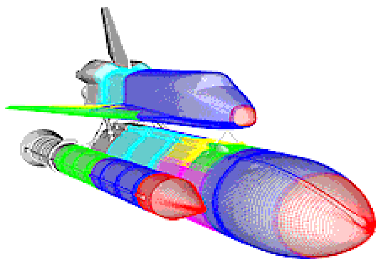




SU.10(FM.1) Fluid Mechanics, Simplified Analyses

Conservation principles, *Eulerian* viewpoint

$$\mathrm{D}M = \frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho \, d\tau + \oint_{\mathrm{CS}} \rho \mathbf{V} \cdot \hat{\mathbf{n}} \, d\sigma = 0$$

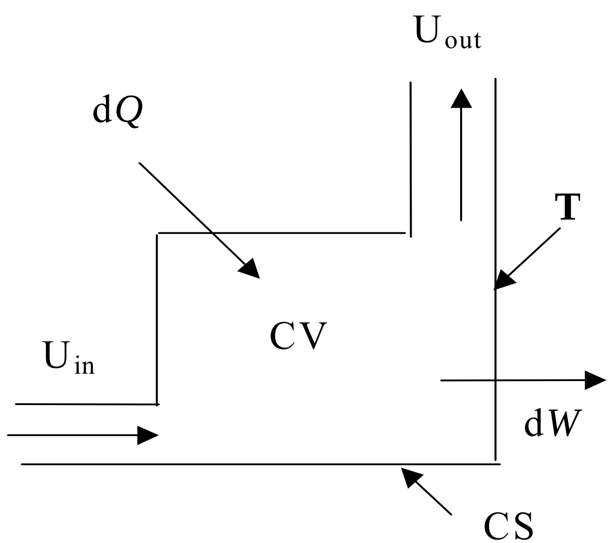


$$\mathrm{D}\mathbf{P} \Rightarrow \frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho \mathbf{V} \, d\tau + \oint_{\mathrm{CS}} \mathbf{V} \rho \mathbf{V} \cdot \hat{\mathbf{n}} \, d\sigma = \int_{\mathrm{CV}} \rho \mathbf{B} \, d\tau + \int_{\mathrm{CS}} \mathbf{T} \, d\sigma$$

$$\mathrm{D}E \Rightarrow \frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho e \, d\tau + \oint_{\mathrm{CS}} \left(e + p/\rho\right) \rho \mathbf{V} \cdot \hat{\mathbf{n}} \, d\sigma = \int_{\mathrm{CV}} s \, d\tau + \oint_{\mathrm{CS}} \left(W - \mathbf{q} \cdot \hat{\mathbf{n}}\right) d\sigma$$

Control volume, uni-directional flow

$\mathrm{D}M$, $\mathrm{D}E$ $\Rightarrow$ algebraic equations
physics $\Rightarrow$ heat added, work done
$\mathrm{D}\mathbf{P}$ $\Rightarrow$ reaction force $\mathbf{T}$



data: velocity in, heat added, fluid properties
output: velocity out, work done, reaction force

SU.11(FM.8) Aerodynamics, Weak Interaction

Farfield, subsonic-transonic potential flow assumption

DM:

$$\mathsf{L}(\phi) = (1 - M_\infty^2)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

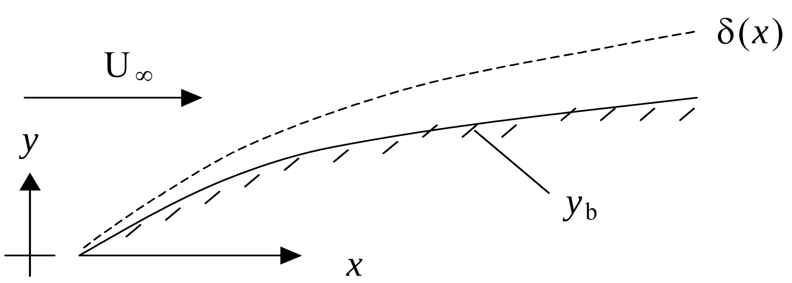
$$\ell(\iota) = \hat{\mathbf{n}} \cdot \nabla\phi - U_\infty \hat{\mathbf{i}} \cdot \hat{\mathbf{n}} = 0$$

DE:

$$p(\mathbf{x}_\delta) = p_\infty - \rho\nabla\phi \cdot \nabla\phi / 2$$

Nearfield, boundary layers wash aerosurfaces

viscous, turbulent effects dominate



DM: $\nabla \cdot \mathbf{u} = 0$

DP: $\mathbf{u} \cdot \nabla\mathbf{u} = \dfrac{-1}{\rho_0}\nabla p + \nabla\mathbf{T}$

in the region $y_b(x) \le y(x) \le \delta(x)$

$\delta(x) \equiv$ boundary layer thickness
$\mathbf{T}$ = viscous + turbulent effects

SU.12(FM.18) Streamfuction-Vorticity Navier-Stokes

For n = 2: $\mathbf{u} = \nabla \times \psi \hat{\mathbf{k}}$ *and* $\omega \equiv \nabla \times \mathbf{u} \cdot \hat{\mathbf{k}}$

DM: $\nabla \cdot \mathbf{u} = \nabla \cdot \nabla \times \psi \hat{\mathbf{k}} = 0$ identically

$\hat{\mathbf{k}} \cdot \nabla \times \mathrm{DP}$: $\omega_t + \nabla \times \psi \hat{\mathbf{k}} \cdot \nabla \omega - \mathrm{Re}^{-1} \nabla^2 \omega = 0$

kinematics: $\omega = \nabla \times \mathbf{u} \cdot \hat{\mathbf{k}} = \nabla \times \nabla \times \psi \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = -\nabla^2 \psi$

Steady-state N-S PDEs + BCs:

$$\mathrm{L}(\omega) = -\mathrm{Re}^{-1} \nabla^2 \omega + \nabla \times \psi \hat{\mathbf{k}} \cdot \nabla \omega = 0$$

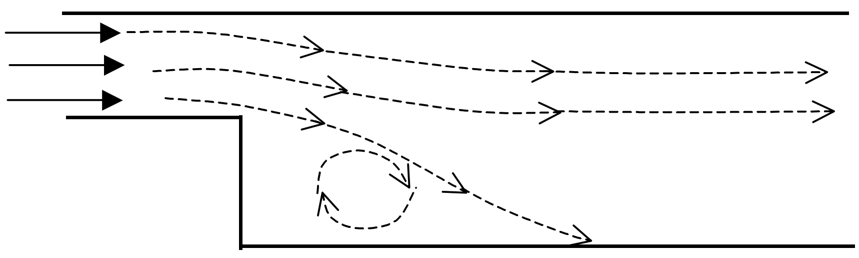
$$\mathrm{L}(\psi) = -\nabla^2 \psi - \omega = 0$$

$\partial\Omega_{\text{in}}$: $\mathbf{u}(y, x_{\text{in}}) \Rightarrow \omega_{\text{in}}, \psi_{\text{in}}$ via definitions

$\partial\Omega_{\text{out}}$: $\hat{\mathbf{n}} \cdot \nabla(\omega, \psi) = 0$

$\partial\Omega_{\text{wall}}$: $\psi = \psi_w$ = constant

$\hat{\mathbf{n}} \cdot \nabla \omega = f_w(\psi, \omega)$



SU.13(ST1.1) Unsteady Scalar Transport

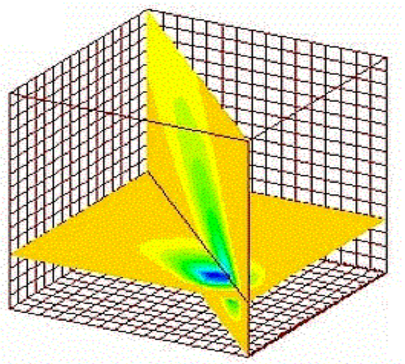
Eulerian non-D description for scalar transport

$$\mathsf{L}(q) = \frac{\partial q}{\partial t} + \mathbf{u}\cdot\nabla q - \mathrm{Pa}^{-1}\nabla\cdot(1+\mathrm{Pa}^{t})\nabla q - s = 0\,, \quad \text{on} \quad \Omega\times t$$

$$\ell(q) = \nabla q\cdot\mathbf{n} + \mathrm{Pb}(q - q_{ref}) + f_n = 0\,, \quad \text{on } \partial\Omega_r \times t$$

$$q(\mathbf{x}_b, t) = q_b(\mathbf{x}_b, t)\,, \quad \text{on } \partial\Omega_b \times t$$

$$q(\mathbf{x}, t_0) = q_0(\mathbf{x})\,, \quad \text{on } \Omega\cup\partial\Omega\times t_0$$



Definitions for (x,t), Pa, Pb, Pa^t depend on application

Transport	q	Pa	Pb	Pa^t
heat	Θ	RePr	Nu	Re^t/Pr^t
mass	Y	ReSc	Pa^{-1}	Re^{t}Sc^t
pollutant	Y_α	ReSc	Pa^{-1}	Re^{t}Sc^t

$$\mathrm{Re}^t \equiv \left(\frac{\nu^t}{\nu}\right)_{\mathrm{dim}}$$

SU.14(STn.13) Unsteady n – D Scalar Transport

Essential ingredients of GWS^h (mDE) + θTS

approximation : $q(\mathbf{x},t) \approx q^N(\mathbf{x},t) \equiv q^h(\mathbf{x},t) = \cup_e q_e(\mathbf{x},t)$

FE basis : $q_e(\mathbf{x},t) = \{N_k(\zeta,\eta)\}^T \{Q(t)\}_e$

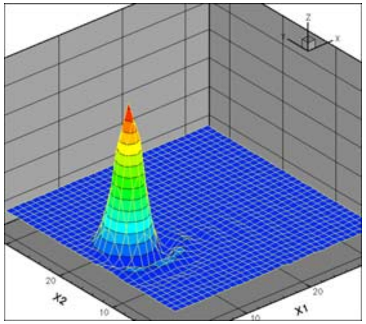
error extremization : $\mathrm{GWS}^N = \int_\Omega \Psi_\beta(\mathbf{x}) \mathrm{L}^m(q^N) d\tau \equiv \{0\} \Rightarrow \mathrm{GWS}^h$

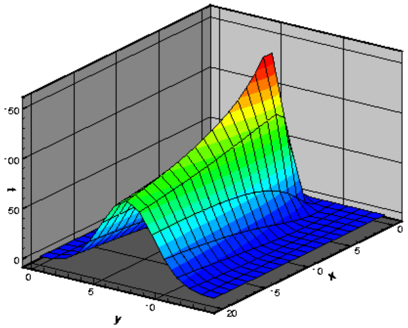
matrix statement : $\mathrm{GWS}^h + \theta\mathrm{TS} \Rightarrow [m\mathrm{JAC}]\{\Delta Q\} = -\Delta t\{m\mathrm{RES}\}$

$[m\mathrm{JAC}] = \mathrm{S}_e([m\mathrm{JAC}]_e), \quad \{m\mathrm{RES}\} = \mathrm{S}_e(\{m\mathrm{RES}\}_e)$

asymptotic convergence : $\|e^h(t)\|_E \le C h_e^{f(k,\mathrm{Pe},\beta)} \|\mathrm{data}\|_{\mathrm{L2}}^2 + C_t \Delta t^{f(\theta)} \|q_0\|_E$

error spectra : $U_\omega \,\&\, G \Rightarrow f(\omega,k,h,\Delta t,\theta,\alpha,\beta,\gamma)$





Template pseudo-code converts theory ⇒ practice

$\mathrm{GWS}^h(m\mathrm{D}E) + \theta\mathrm{TS} \Rightarrow \mathrm{S}_e\{\mathrm{WS}\}_e \equiv \{0\}$

$\{\mathrm{WS}\}_e \equiv (\mathrm{const})(\mathrm{avg})_e\{\mathrm{dist}\}_e(\mathrm{metric})[\mathrm{Matrix}]_e\{Q \text{ or } \mathrm{data}\}_e$

$[\mathrm{JAC}]_e \equiv \partial\{\mathrm{WS}\}_e / \partial\{Q\}_e$