CD.1 Unsteady Convection-Diffusion

Energy transport with boundary convection, *n* = 1

DE:

$$\mathrm{L}(T) = \rho c_p \frac{\partial T}{\partial t} + \rho c_p u \frac{\partial T}{\partial x} - \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) - s = 0 \quad \text{, on } \Omega \times t \subset \Re^n \times \Re^1$$

BCs:

$$\ell(T) = k \frac{\mathrm{d}T}{\mathrm{d}x} + h(T - T_r) + f_n = 0 \quad \text{, on } \partial\Omega_R \times t$$

$$T(x_b, t) = T_b(x_b, t) \quad \text{, on } \partial\Omega_D \times t$$

Galerkin weak statement process

approximation

$$T(x,t) \cong T^N(x,t) \equiv \sum_{\alpha}^{N} \Psi_\alpha(x) Q_\alpha(t)$$

$$\mathrm{GWS}^N \equiv \int_\Omega \Psi_\beta \mathrm{L}(T^N)\mathrm{d}x = \int_\Omega \Psi_\beta \left(\frac{\partial T^N}{\partial t} + u \frac{\partial T^N}{\partial x} - \kappa \frac{\partial^2 T^N}{\partial x^2} - s \right) \mathrm{d}x$$

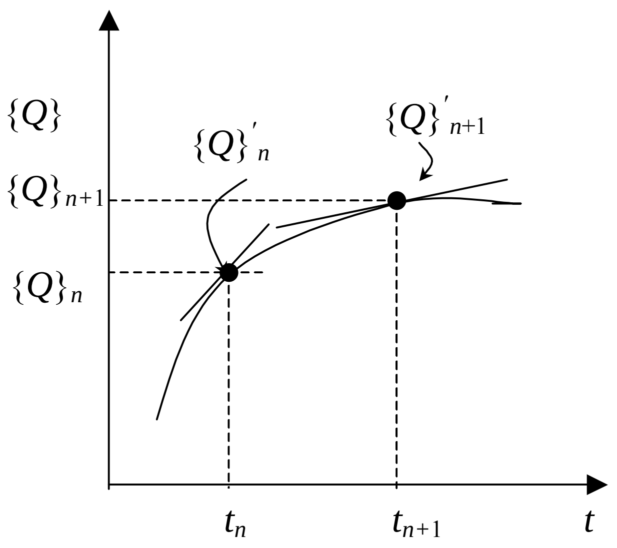
FE implementation $T^N(x,t) = \cup_e T_e(x,t), \quad T_e(x,t) \equiv \{N_k(\zeta)\}^T \{Q(t)\}_e$

$$\mathrm{GWS}^h = \mathrm{S}_e\left([\mathrm{MASS}]_e \frac{\mathrm{d}\{Q\}_e}{\mathrm{d}t} + \left([\mathrm{UVEL}]_e + [\mathrm{DIFF}]_e + [\mathrm{HBC}]_e \right) \{Q\}_e - \{\mathrm{b}\}_e \right)$$

CD.2 Unsteady GWS^h ODE System Utilization

GWS^h has produced an ODE system

$$\mathrm{GWS}^h \Rightarrow \{Q\}' = -[\mathrm{MASS}]^{-1}\{\mathrm{RES}\}$$



Time Taylor series

$$\{Q\}_{n+1} = \{Q\}_n + \Delta t\left(\theta\{Q\}'_{n+1} + (1-\theta)\{Q\}'_n\right) + O\left(\Delta t^{f(\theta)}\right)$$

$$\text{implicitness parameter: } 0 \le \theta \le 1.0$$

Substituting GWS^h into θTS

$$\{Q\}_{n+1} = \{Q\}_n - \Delta t[\mathrm{MASS}]^{-1}(\theta\{\mathrm{RES}\}_{n+1} + (1-\theta)\{\mathrm{RES}\}_n) + \mathrm{TE}$$

$$\text{clearing unknown, } \{\Delta Q\} \equiv \{Q\}_{n+1} - \{Q\}_n$$

$$([\mathrm{MASS}] + \theta\Delta t[\mathrm{UVEL} + \mathrm{DIFF} + \mathrm{HBC}])\{\Delta Q\} = -\Delta t\{\mathrm{RES}\}_n$$

$$\{\mathrm{RES}\}_n = [\mathrm{UVEL} + \mathrm{DIFF} + \mathrm{HBC}]\{Q\}_n - \{\mathrm{b}(data)\}_{n+\theta}$$

CD.3 GWS^h + θTS Template, n = 1

Summary: GWS^h + θTS for heat conduction

DE:

$$\mathrm{L}(T) = T_t + uT_x - \kappa T_{xx} - s = 0$$
$$\ell(T) = T_n + h(T - T_r) + f_n = 0$$

$$\mathrm{GWS}^h + \theta\mathrm{TS} \Rightarrow \mathrm{S}_e\{\mathrm{WS}\}_e = \{0\} \Rightarrow [\mathrm{JAC}]\{\Delta Q\} = -\Delta t\{\mathrm{RES}\}_n$$
$$[\mathrm{JAC}]_e = [\mathrm{MASS}]_e + \theta\Delta t[\mathrm{UVEL} + \mathrm{DIFF} + \mathrm{HBC}]_e$$
$$\{\mathrm{RES}\}_e = [\mathrm{UVEL} + \mathrm{DIFF} + \mathrm{HBC}]_e\{Q\}_n - \{\mathrm{b}(s, T_r, h, f_n)\}_{n+\theta}$$

Template pseudo-code:

$$\{\mathrm{WS}\}_e = (\mathrm{const})(\mathrm{avg})_e\{\mathrm{dist}\}_e^T(\mathrm{metric})_e[\mathrm{Matrix}]\{Q \text{ or data}\}_e$$

$$\begin{aligned}
[\mathrm{JAC}]_e = {} & (\ )(\ )\{\ \}(1)[\mathrm{A200}][\ ] + (\theta\Delta t)(\ )\{\mathrm{U}\}(0)[\mathrm{A3001}][\ ] \\
& + (\theta\Delta t, \kappa)(\ )\{\ \}(-1)[\mathrm{A211}][\ ] \\
& + (\theta\Delta t, \kappa, h)(\ )\{\ \}(\ )[\mathrm{ONE}][\ ]
\end{aligned}$$

$$\begin{aligned}
\Delta t\{\mathrm{RES}\}_e = {} & (\Delta t)(\ )\{\mathrm{U}\}(0)[\mathrm{A3001}]\{\mathrm{QN}\} + (\Delta t, \kappa)(\ )\{\ \}(-1)[\mathrm{A211}]\{\mathrm{QN}\} \\
& + (\Delta t, \kappa, h)(\ )\{\ \}[\mathrm{ONE}]\{\mathrm{QN}\} + (-\Delta t)(\ )\{\ \}(1)[\mathrm{A200}]\{\mathrm{SRC}\} \\
& + (-\Delta t, \kappa, h)(\ )\{\ \}(\ )[\mathrm{ONE}]\{\mathrm{QR}\}
\end{aligned}$$

CD.4 Error Estimates, n = 1 Unsteady GWS^h + θTS

For *any* solution $q^h(x, t)$ for unsteady L(q)

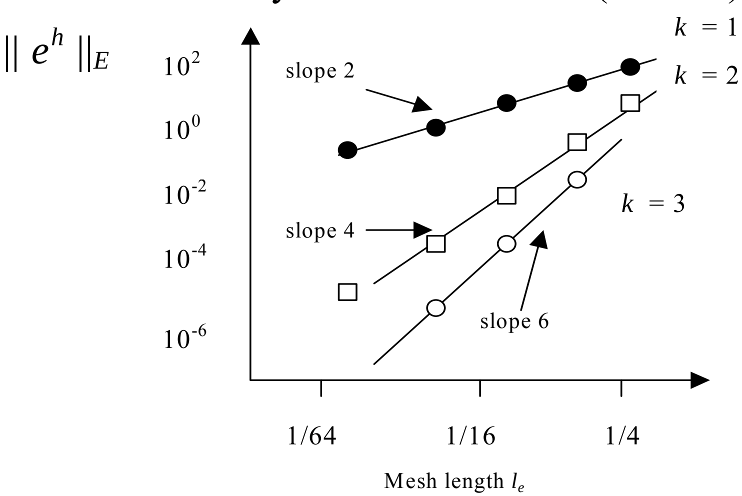
$$e^h(x, t) \equiv q(x, t) - q^h(x, t)$$

Asymptotic error estimates are f (κ, {N_k}, θ, data)

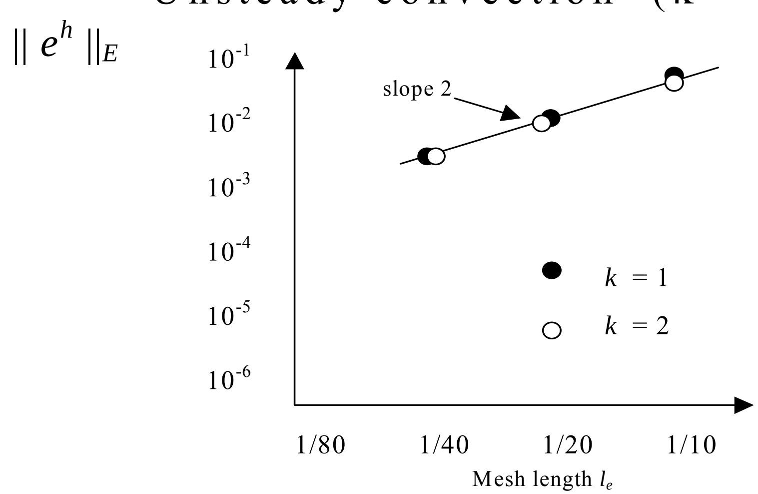
$$\kappa > 0 \quad : \left\| e^h(t) \right\|_E \leq C \ell_e^{2\gamma} \left\| \text{data} \right\|_{L2}^2 + C_t \Delta t^{f(\theta)} \left\| q_0 \right\|_E \;, \gamma = \min(k, r-1)$$

$$\kappa = 0 \quad : \left\| e^h(t) \right\|_E \leq C \ell_e^2 \int_{t_0}^{t} \left\| q(x,\tau) \right\|_{H^{k+1}}^2 d\tau + C_t \Delta t^{f(\theta)} \left\| q_0 \right\|_E$$

Unsteady conduction (κ > 0)



Unsteady convection (κ = 0)



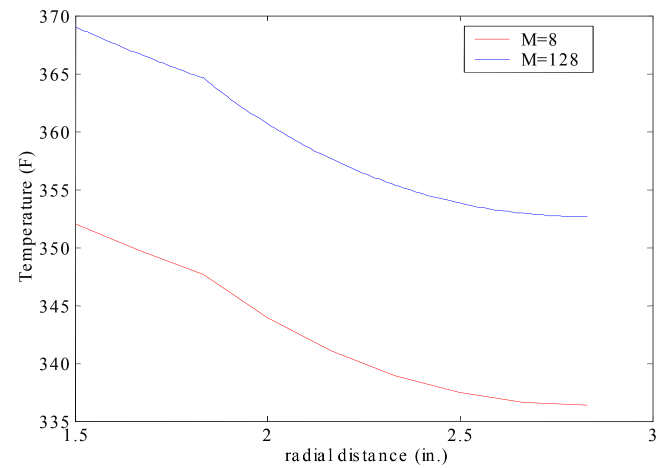
CD.5 Finned Cylinder, Data Smoothness

Problem statement

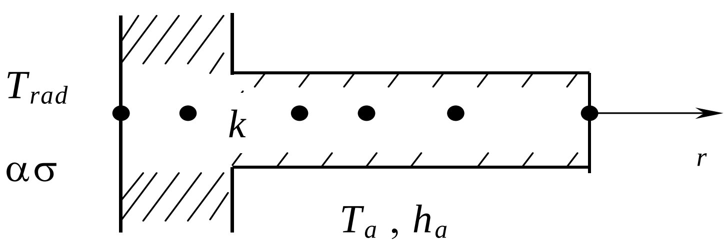
$$L(T) = -\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0$$

$$\ell(T) = k\,dT/dn + h(T - T_{\alpha}) + \alpha\sigma(T^4 - T_r^4) = 0$$

Solution graph



Data



Error estimate:

$$\left\|e^h\right\|_E \le C l_e^{2\gamma} \left\|\text{data}\right\|_{L2}^2 ,\ \gamma = \min(k, r-1)$$

Energy norm convergence data, *k* = 1

Mesh	M	$\|\|T\|\|_E$	slope
h	8	2.36494E+03	
$h/2$	16	2.42266E+03	
$h/4$	32	2.45369E+03	0.89486
$h/8$	64	2.46981E+03	0.94566
$h/16$	128	2.47802E+03	0.97236

Energy norm convergence, [HBC] on all elements

Mesh	M	$\|\|T\|\|_E$	slope
h	8	2.28175700E+03	
$h/2$	16	2.28230949E+03	
$h/4$	32	2.28244772E+03	1.99890
$h/8$	64	2.28248228E+03	1.99972
$h/16$	128	2.28249092E+03	1.99993

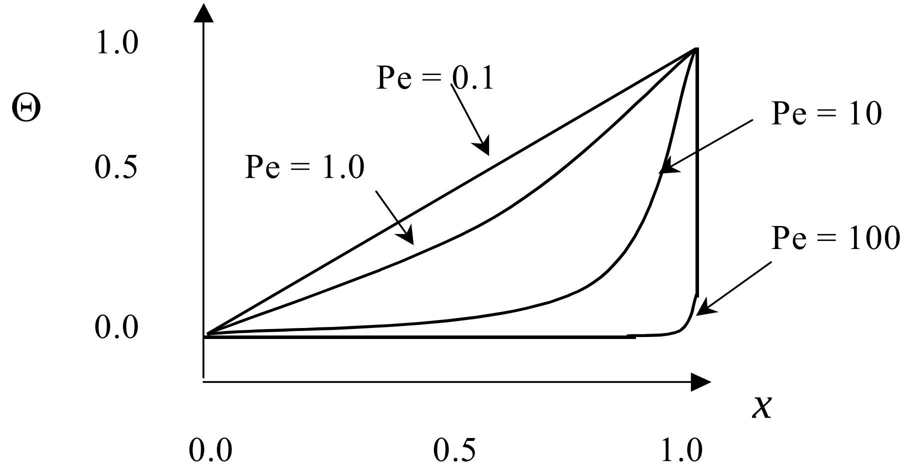
CD.6 Peclet Problem, Dispersion Error

Problem statement

DE: $L(\Theta) = \frac{d\Theta}{dx} - \frac{1}{Pe}\frac{d^2\Theta}{dx^2} = 0$

BCs: $\Theta(0) = 0, \quad \Theta(1) = 1$

Solution graph



Analytical solution

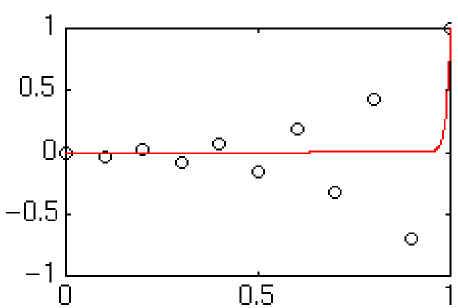
$$\Theta(x) = \frac{1 - \exp(x\,Pe)}{1 - \exp(Pe)}$$

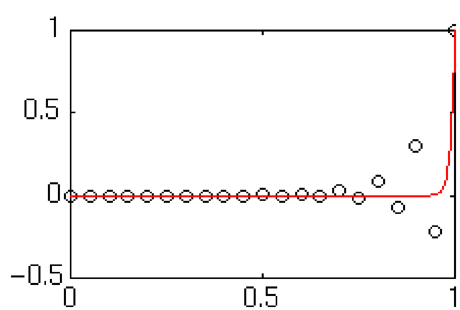
GWS^h solutions, Pe = 10^2

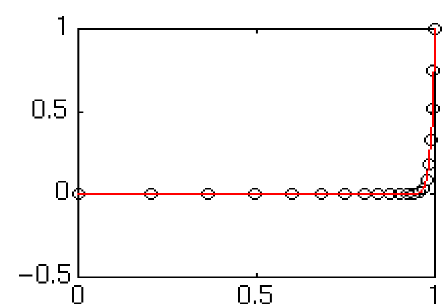
uniform Ω^h, M = 10, k = 1

uniform Ω^h, M = 10, k = 2

non-uniform Ω^h, M = 20, k = 1, 2







CD.7 Mass Transport, Dispersion, Diffusion

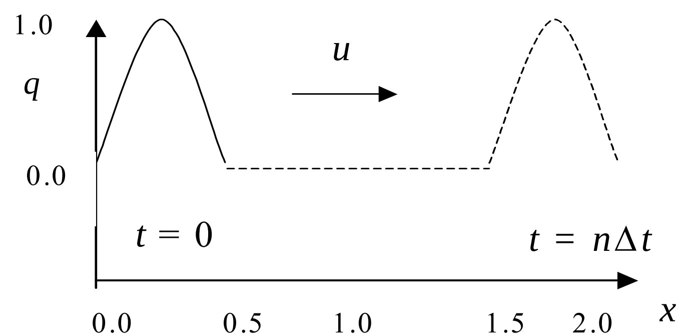
Problem statement

DM: $\mathrm{L}\,(q) = \dfrac{\partial q}{\partial t} + u\,\dfrac{\partial q}{\partial x} = 0$

BC: $q(x = 0,\, t) = 0$

IC: $q(x,\, t = t_0) = q_0\,(x)$

Solution graph



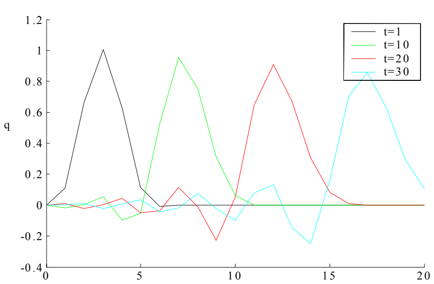
Courant No : $\mathrm{C} \equiv u \Delta t / \Delta x$

Analytical (characteristic) solution

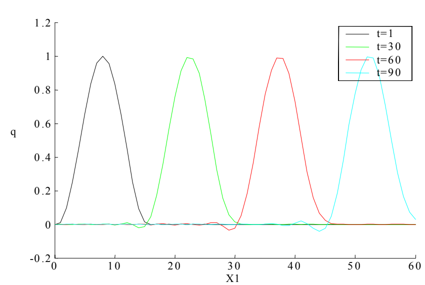
$$q(x,\, t) = q_0 \exp i(x - ut)$$

GWS^h + θTS solutions, k = 1 basis

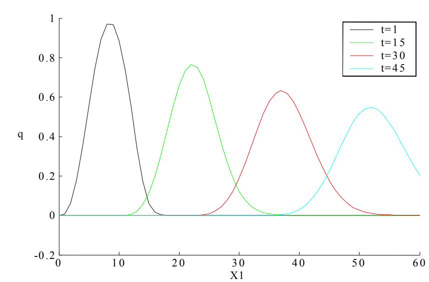
M = 20, C = 0.5 = θ



M = 60, C = 0.5 = θ



M = 60, C = 1.0 = θ



CD.8 Unsteady Conduction, Computer Lab #2

Solution-adaptive meshing $\Rightarrow$ *accurate* GWS^h + θTS solution

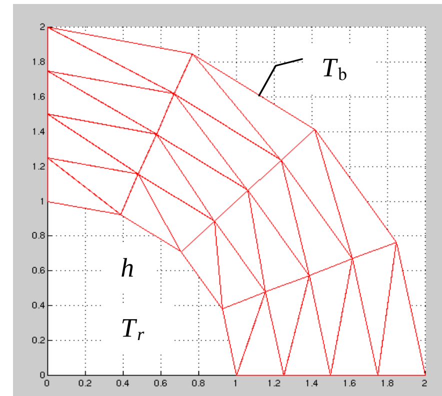
DE: $\mathrm{L}(T) = \dfrac{\partial T}{\partial t} - \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(\kappa r \dfrac{\partial T}{\partial r}\right) = 0$

BCs: $\ell(T) = \kappa \mathrm{d}T/\mathrm{d}r + h\,(T - T_r) = 0$

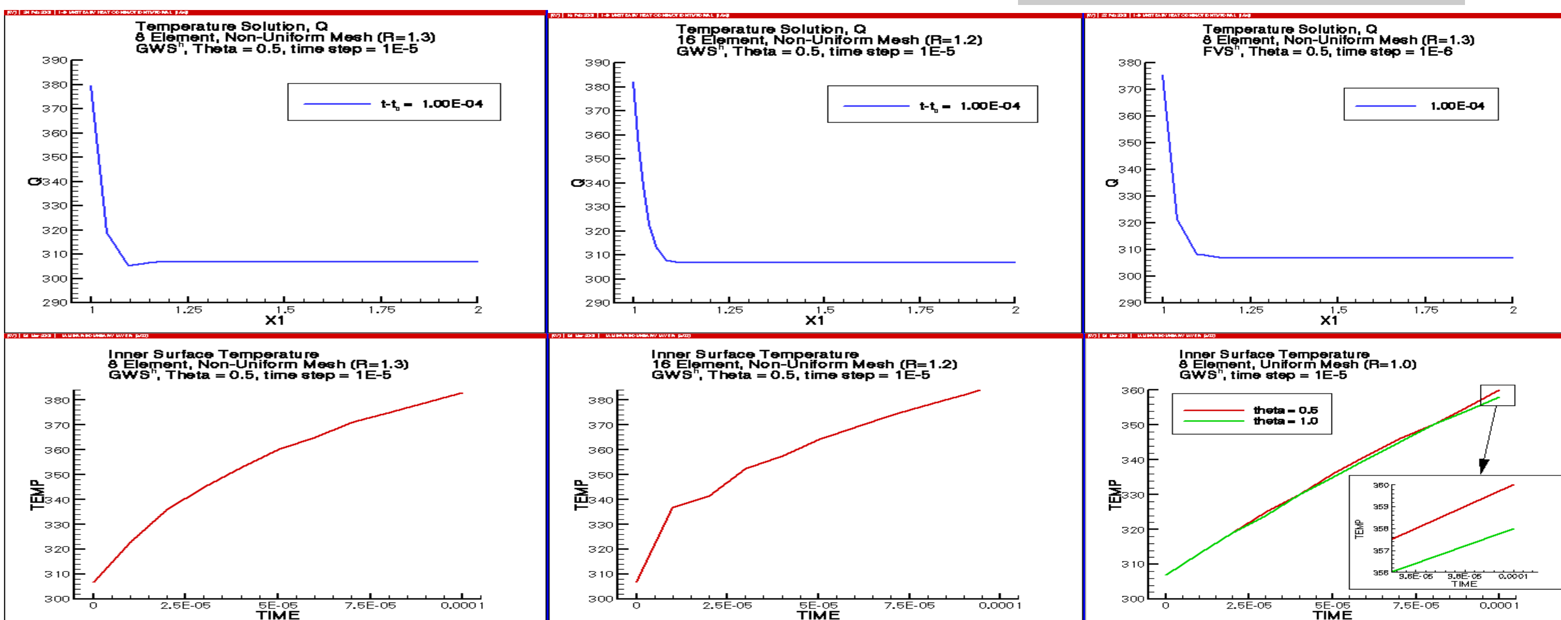
$T(r_b) = T_b$

IC: $T(r, t_0) \equiv T_0 << T_r$

Problem geometry



Solution summary



CD.9 GWS^h - FVS^h Algorithm Comparison

GWS^h + θTS *aPSE* Newton template, L(T) in polar coordinates

$$\{\mathrm{F}Q\}_e = (\quad)(\quad)\{\mathrm{X1}\}(0;1)[\mathrm{A3000}]\{QP - Q\mathrm{N}\}$$
$$+ (\Delta t)(\quad)(\mathrm{U1})(0;0)[\mathrm{A3001}]\{QP + Q\mathrm{N}\}_\theta$$
$$+ (\Delta t, \mathrm{COND})(\ )\{\mathrm{X1}\}(0;-1)[\mathrm{A3011}]\{QP + Q\mathrm{N}\}_\theta$$
$$+ (\Delta t, \mathrm{HL}, \mathrm{RL})(\ )\{\ \}(0;0)[\mathrm{ONE}]\{QP - \mathrm{TR}\}_\theta$$
$$[\mathrm{JAC}]_\mathrm{e} = \partial\{\mathrm{F}Q\}_e / \partial\{\mathrm{Q}P\}_e$$

Sole modification for FVS^h + θTS algorithm (Ch.3.7)

$$\mathrm{GWS}^h : \int_{\Omega_e} \{N\}\{N\}^T r\mathrm{d}r \{QP - Q\mathrm{N}\}_e \Rightarrow \ell_e \{\mathrm{R}\}_e^T [\mathrm{A3000}]\{QP - Q\mathrm{N}\}_e$$

$$\mathrm{FVS}^h : \int_{\Omega_v} \frac{\partial T}{\partial t} r\mathrm{d}r \Rightarrow r^2/2\Big|_L^R (Q_j^{n+1} - Q_j^n)/\Delta t = \overline{\mathrm{R}}_j \ell_v (Q\mathrm{P} - Q\mathrm{N})_j$$

$$\text{for } \{N_1\} : \mathrm{GWS}_e \approx \ell_e \overline{\mathrm{R}}_e [\mathrm{A200}], \quad [\mathrm{A200}] = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathrm{FVS}_e \equiv \ell_v \overline{\mathrm{R}}_e [\mathrm{A200F}], \quad [\mathrm{A200F}] = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathrm{S}_j(\cdot) \Rightarrow \mathrm{Q.E.D}$$