FSNS.1 Free-Surface Hydrostatic Fluid Mechanics

Tidal, unsteady free-surface flows described by Reynolds-averaged INS

$$\mathrm{D}M: \ \nabla \cdot \mathbf{u} = 0$$

$$\mathrm{D}\mathbf{P}: \ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho_0}\frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_j}\left(\nu + \nu^t\right)\frac{\partial u_i}{\partial x_j} + \frac{\rho g}{\rho_0}\hat{\mathbf{g}}_i + \Omega_{ij} u_j = 0, \quad 1 \le i \le 2$$

$$\mathrm{DP}_z: \ \frac{\partial p}{\partial z} + \rho g = 0$$

$$\mathrm{D}E: \ \frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\kappa + \kappa^t\right)\frac{\partial T}{\partial x_j} - s = 0$$

$$\mathrm{D}M_s: \ \frac{\partial S}{\partial t} + u_j \frac{\partial S}{\partial x_j} - \frac{\partial}{\partial x_j}\left(k_s + k_s^t\right)\frac{\partial S}{\partial x_j} = 0$$

state: $\rho = (\alpha + 0.698\mathrm{R})/\mathrm{R}$, $\quad \alpha = 1780 + 11.25T - 0.07T^2 - (3.8 - 0.01T)S$

$\mathrm{R} = 5890 + 38T - 0.375T^2 + 3S$ (mks units)

S = salinity (parts/thousand)

Coriolis: $\Omega_{ij} = -2\omega \sin\phi\, \mathrm{e}_{ij}$, $\quad \omega$ = earth rotation rate, ϕ = latitude

e_{ij} = cartesian alternator

FSNS.2 Free-Surface INS, PDE + BCs Well-Posedness

FSNS conservation principles form is ill-posed EBV

$$\mathrm{DP}_z \;= \frac{\partial w}{\partial t} + u_j \frac{\partial w}{\partial x_j} + \cdots \text{ was discarded as higher} - \text{order}$$

$$\mathrm{D}M \Rightarrow \frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \quad \text{is not a well} - \text{posed ODE for either} \pm \hat{\mathbf{k}} \text{ integration}$$

Well-posed resolution is to append H.O. DP_z and migrate to PPNS algorithm

$$\mathrm{D}M^h \Rightarrow \mathcal{L}(\phi) = -\nabla^2\phi - \nabla \cdot \mathbf{u} = 0$$

$$P^*_{n+1} = \sum \Phi + (\theta \Delta t)^{-1} \sum_{\alpha=0}^{p} \phi^{\alpha+1}$$

$$\text{in } \mathrm{D}\mathbf{P}: \; \frac{1}{\rho_0} p \Rightarrow P^* + p/\rho_0$$

$$\int \mathrm{DP}_z: \; p = p_{\mathrm{atm}} + \gamma g (z - z_s)$$

Closure for turbulence is typically the *k*-ε model

$$\mathrm{D}E': \; \frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} - \frac{\partial}{\partial x_j}\left((\nu + \nu^t) \frac{\partial k}{\partial x_j} \right) - \tau_{ij} \frac{\partial u_i}{\partial x_j} + \varepsilon = 0$$

$$\mathrm{D}E'': \; \frac{\partial \varepsilon}{\partial t} + u_j \frac{\partial \varepsilon}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\mathrm{C}_\varepsilon \nu^t \frac{\partial \varepsilon}{\partial x_j} \right) - \mathrm{C}^1_e \frac{k}{\varepsilon} \tau_{ij} \frac{\partial u_i}{\partial x_j} + \mathrm{C}^2_\varepsilon \frac{\varepsilon^2}{k} = 0$$

FSNS.3 Depth-Averaged Hydrostatic Free-Surface Flows

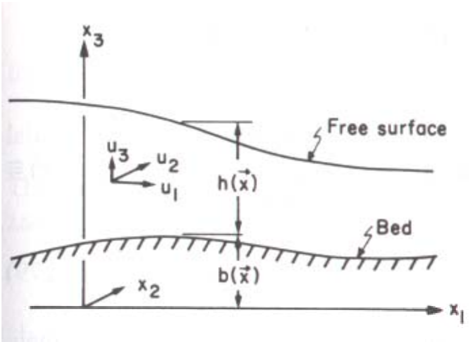
In many instances, depth-averaged description suffices

data : $\mathrm{b}(x_1, x_3) \Rightarrow$ bed profile (no errosion)

$h(x_1, x_2, t) \Rightarrow$ flow depth

$\zeta = h - h_0 \Rightarrow$ free surface elevation

$h_0 =$ mean flow depth



Depth-averaged velocity definition

$$\bar{u}_i(x_1, x_2, t) \equiv \frac{1}{h}\int_{\mathrm{b}}^{h+\mathrm{b}} u_i(x_1, x_2, x_3, t)\mathrm{d}x_3$$

Hydrostatic free-surface INS, depth-averaged partially-parabolic PDE system

$$\mathrm{D}M : \ \mathcal{L}(h) = \frac{\partial h}{\partial t} + \frac{\partial m_j}{\partial x_j} = 0 \quad , \ m_j \equiv h\bar{u}_j$$

$$\mathrm{D}\mathbf{P} : \ \mathcal{L}(m_i) = \frac{\partial m_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u}_j m_i + \frac{1}{2}gh^2\delta_{ij} - h\tau_{ij}\right) + hg\frac{\partial \mathrm{b}}{\partial x_i} - \tau_{ij}\frac{\partial h}{\partial x_j} + \Omega_{ij}m_j - \tau_{is} + \tau_{ib} = 0$$

FSNS.4 Depth-Averaged FS INS, Closure Models, BCs

Stress tensor τ_{ij} closure models

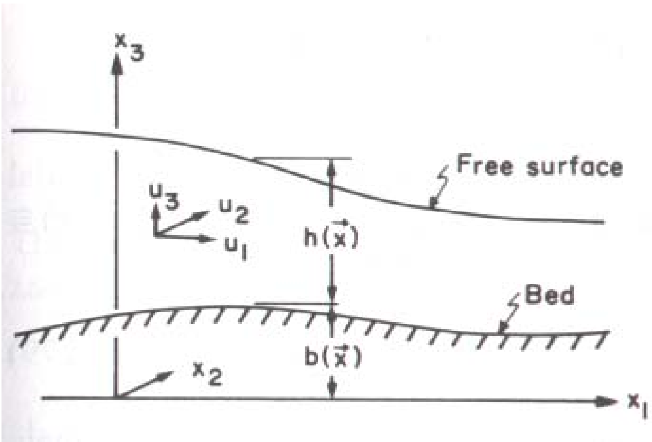
$$\text{flowfield: } \tau_{ij} = (\nu + \nu^t)\overline{\mathrm{S}}_{ij} \quad , \quad \overline{\mathrm{S}}_{ij} \equiv \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}$$

$$\text{free surface: } \tau_{is} = \frac{\gamma}{\rho} w_i (w_j w_j)^{1/2} \quad , \quad w_j = \text{wind velocity vector}$$

$$\text{bed surface: } \tau_{ib} = -g\overline{u}_i \mathrm{C}^{-2} (\overline{u}_j \overline{u}_j)^{1/2}, \ \mathrm{C} = \text{Chezy bed drag coefficient}$$

Well-posedness, boundary conditions, sub-critical flow

$$\text{on } \partial\Omega_{in} \ : \overline{\mathbf{u}}(x_s, t) = \text{data}$$
$$\mathbf{m} \cdot \hat{\mathbf{n}} = \text{not specifiable}$$
$$\mathbf{m} \cdot \hat{\mathbf{s}} = \text{data}$$
$$\text{on } \partial\Omega_{\text{out}} \ : \hat{\mathbf{n}} \cdot \nabla \mathbf{m} = \mathbf{0}$$
$$h = \text{data}$$
$$\text{on } \partial\Omega_{\text{farfield}} : \hat{\mathbf{n}} \cdot \nabla q = 0$$
$$\text{on } \partial\Omega_{\text{walls}} \ : \mathbf{m} = \mathbf{0}$$
$$\hat{\mathbf{n}} \cdot \nabla h = 0 \text{ implied}$$



FSNS.5 Depth-Averaged FS INS, Non-Dimensionalization

Depth-averaged, free-surface INS partially-parabolic non-D PDE system

$$\mathrm{D}M : \mathcal{L}(h) = \frac{\partial h}{\partial t} + \beta_1 \frac{\partial m_j}{\partial x_j} = 0$$

$$\mathrm{D}\mathbf{P} : \mathcal{L}(m_i) = \frac{\partial m_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u}_j m_i - \frac{1}{\mathrm{Re}}\left(1 + \mathrm{Re}^t\right)\mathrm{S}_{ij} + \frac{1}{2\mathrm{Fr}^2} h^2 \delta_{ij} \right) + \frac{1}{\mathrm{Fr}_\mathrm{b}^2} h \frac{\partial \mathrm{b}}{\partial x_i} + \frac{1}{\mathrm{Fr}_\mathrm{D}^2}\frac{h^2}{2}\frac{\partial \rho}{\partial x_i} + \mathrm{Co}\ \mathrm{e}_{ij}\bar{u}_j - \frac{1}{\mathrm{Re}}\left(\tau_{is} - \tau_{ib}\right) = 0$$

$$\mathrm{D}E' : \mathcal{L}(k) = \frac{\partial k}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u}_j k - \frac{1}{\mathrm{Re}}\left(\frac{1}{\mathrm{Pr}} + \frac{\mathrm{Re}^t}{\mathrm{Pr}_t} \right)\frac{\partial k}{\partial x_j} \right) - \tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} + \varepsilon = 0$$

$$\mathrm{D}E'' : \mathcal{L}(\varepsilon) = \frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u}_j \varepsilon - \frac{1}{\mathrm{Re}}\left(\frac{1}{\mathrm{Pr}} + \frac{\mathrm{C}_\varepsilon \mathrm{Re}^t}{\mathrm{Pr}^t} \right)\frac{\partial \varepsilon}{\partial x_j} \right) - \mathrm{C}_\varepsilon^1 \frac{k}{\varepsilon}\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} + \mathrm{C}_\varepsilon^2 \frac{\varepsilon^2}{k} = 0$$

$$\mathrm{D}\tau_{ij} : \mathcal{L}(\tau_{ij}) = \tau_{ij} - \frac{2}{3}k\delta_{ij} + \left(\mathrm{Re}^t / \mathrm{Re}\right)\left(\partial \bar{u}_i / \partial x_j + \partial \bar{u}_j / \partial x_i\right) = 0$$

reference length scales: $\mathrm{L_r}$ = transverse span, $\mathrm{H_r}$ = vertical span, ζ_r = flow depth

non-D groups:

- β_1 = relative depth scale $= \mathrm{H_r}/\zeta_\mathrm{r}$
- β_2 = relative length scale $= \mathrm{H_r}/\mathrm{L_r}$
- Fr = depth scale Froude number $= \mathrm{U_r}/\sqrt{g\mathrm{H_r}}$
- $\mathrm{Fr_b}$ = length scale Froude number $= \mathrm{U_r}/\sqrt{g\mathrm{L_r}}$
- $\mathrm{Fr_D}$ = densimetric Froude number $= \mathrm{Fr}/\varepsilon_0$, $\varepsilon_0 = (\rho_\mathrm{r} - \rho_0)/\rho_0$
- Co = Coriolis number $= \mathrm{fL_r}/\mathrm{U_r}$
- Re = Reynolds number $= \mathrm{U_r L_r}/\upsilon$
- Pr = Prandtl number $= c_p \upsilon / k$
- Re^t = turbulent Reynolds number $= (\upsilon^t/\upsilon)_\mathrm{dim}$

FSNS.6 Free-Surface INS, Non-Dimensional PDE System

Pressure projection free-surface hydrostatic flow PDE system

$$\mathrm{D}M^{h}:\ \mathcal{L}(\phi) = -\nabla^{2}\phi - \nabla\cdot\mathbf{u} = 0$$

$$\mathrm{D}\mathbf{P}:\ \mathcal{L}(u_i) = \frac{\partial u_i}{\partial t} + u_j\frac{\partial u_j}{\partial x_j} - \frac{1}{\mathrm{Re}}\frac{\partial}{\partial x_j}\left(\left(1+\mathrm{Re}^{t}\right)\frac{\partial u_i}{\partial x_j}\right) + \frac{1}{\mathrm{Fr}_{\mathrm{D}}^{2}}\frac{\partial\zeta}{\partial x_i}(1-\delta_{i3}) + \frac{\partial(\mathrm{P}^{*} + p/\rho_0)}{\partial x_i} - \mathrm{Co}\ \mathrm{e}_{ij}u_t = 0$$

$$\mathrm{D}E:\ \mathcal{L}(\Theta) = \frac{\partial\Theta}{\partial t} + u_j\frac{\partial\Theta}{\partial x_j} - \frac{1}{\mathrm{Re\,Pr}}\frac{\partial}{\partial x_j}\left(\left(1+\mathrm{Re}^{t}\right)\frac{\partial\Theta}{\partial x_j}\right) - s_{\Theta} = 0$$

$$\mathrm{D}M_s:\mathcal{L}(S) = \frac{\partial S}{\partial t} + u_j\frac{\partial S}{\partial x_j} - \frac{1}{\mathrm{Re\,Sc}}\frac{\partial}{\partial x_j}\left(\left(1+\mathrm{Re}^{t}\right)\frac{\partial S}{\partial x_j}\right) = 0$$

$$\mathrm{D}E':\ \mathcal{L}(k) = \frac{\partial k}{\partial t} + u_j\frac{\partial k}{\partial x_j} - \frac{1}{\mathrm{Re}}\left(\frac{\partial}{\partial x_j}\left(\frac{1}{\mathrm{Pr}} + \frac{\mathrm{Re}^{t}}{\mathrm{Pr}^{t}}\right)\frac{\partial k}{\partial x_j}\right) - \tau_{ij}\frac{\partial u_i}{\partial x_j} + \varepsilon = 0$$

$$\mathrm{D}E'':\ \mathcal{L}(\varepsilon) = \frac{\partial\varepsilon}{\partial t} + u_j\frac{\partial\varepsilon}{\partial x_j} - \frac{1}{\mathrm{Re}}\left(\frac{\partial}{\partial x_j}\left(\frac{1}{\mathrm{Pr}} + \mathrm{C}_{\varepsilon}\frac{\mathrm{Re}^{t}}{\mathrm{Pr}^{t}}\right)\frac{\partial\varepsilon}{\partial x_j}\right) - \mathrm{C}_{\varepsilon}^{1}\frac{k}{\varepsilon}\tau_{ij}\frac{\partial u_i}{\partial x_j} + \mathrm{C}_{\varepsilon}^{2}\frac{\varepsilon^{2}}{k} = 0$$

$$\text{closure}:\mathbf{P}^{*} = \sum\Phi + (\theta\Delta t)^{-1}\sum_{\alpha=0}^{p}\phi^{\alpha+1}$$

$$p = p_{atm} + \gamma g(z - z_s)$$

comment : reduces to PPNS statement sans "genuine" $\mathcal{L}(p)$!

FSNS.7 Divergence Form for Depth-Averaged FS INS

Sole new conservation law statement is depth-averaged DM + DP

$$\text{state variable}: \; q(x_i, t) = \{h, m_1, m_2, k, \varepsilon, \tau_{ij}\}^T$$

$$\text{divergence form}: \; \mathcal{L}(q) = \frac{\partial q}{\partial t} + \frac{\partial}{\partial x_j}\left(f_j - f_j^v\right) - s = 0$$

$$\text{flux vectors}: \; f_j = \begin{Bmatrix} \beta_1 m_i \\ \dfrac{m_i m_j}{h} + \dfrac{1}{2\mathrm{Fr}^2} h^2 \delta_{ij} \\ m_j k / h \\ m_j \varepsilon / h \end{Bmatrix}, \quad f_j^v = \begin{Bmatrix} 0 \\ \dfrac{1}{\mathrm{Re}}\left(1 + \mathrm{Re}^t\right)\mathrm{S}_{ij} \\ \dfrac{1}{\mathrm{Re\,Pr}}\left(1 + \mathrm{Re}^t\right)\dfrac{\partial k}{\partial x_j} \\ \dfrac{\mathrm{C}_t}{\mathrm{Re\,Pr}}\left(1 + \mathrm{Re}^t\right)\dfrac{\partial \varepsilon}{\partial x_j} \end{Bmatrix}$$

$$\text{data}: \quad \bar{u}_j \equiv m_j / h$$

$$s = \{0, \; f_i(q, \mathrm{b}, \rho, \bar{u}_i, \tau_{ib,s}), \; f_{k,\varepsilon}(\tau_{ij}\bar{u}_i)\}^T$$

$$\mathrm{Re}^t = \left(\upsilon^t / \upsilon\right)_{\mathrm{dim}}$$

$$\mathrm{S}_{ij} = \frac{\partial m_i}{\partial x_j} + \frac{\partial m_j}{\partial x_i}$$

FSNS.8 GWS^h + θTS for Depth-Averaged FS INS

Approximation:

$$q(\mathbf{x},t) \approx q^N(\mathbf{x},t) = \sum_{\alpha=1}^{N} \Psi_\alpha(\mathbf{x}) Q_\alpha(t)$$

$$\mathrm{GWS}^N = \int_\Omega \Psi_\beta(\mathbf{x}) \mathcal{L}(q^N) \mathrm{d}\tau \equiv 0, \ \forall\beta$$

$$= \int_\Omega \Psi_\beta \left[\frac{\partial q^N}{\partial t} + \frac{\partial}{\partial x_j}\left(f_j - f_j^v\right)^N - s \right] \mathrm{d}\tau$$

$$= \int_\Omega \Psi_\beta \left(\frac{\partial q^N}{\partial t} - s \right) \mathrm{d}\tau - \int_\Omega \frac{\partial \Psi_\beta}{\partial x_j}\left(f_j - f_j^v\right)^N \mathrm{d}\tau + \oint_{\partial\Omega} \Psi_\beta \left(f_j - f_j^v\right)^N \hat{\mathbf{n}}_j \mathrm{d}\sigma$$

$$= [\mathrm{MASS}] \frac{\mathrm{d}\{Q\}}{\mathrm{d}t} + \{\mathrm{RES}\} = \{0\}$$

$$\mathrm{GWS}^N + \theta\mathrm{TS} \Rightarrow \{\mathrm{F}Q\} = [\mathrm{MASS}]\{\Delta Q\} + \Delta t \{\mathrm{RES}\}_\theta = \{0\}$$

Finite element implementation:

$$q^N(\mathbf{x},t) \equiv q^h = \cup_e q_e(\mathbf{x},t), \ \ q_e(\mathbf{x},t) \equiv \{N_k(\zeta,\eta)\}^T \{Q(t)\}_e$$

$$\mathrm{GWS}^h + \theta\mathrm{TS} \Rightarrow \{\mathrm{F}Q\} = \mathrm{S}_e\{\mathrm{F}Q\}_e = \{0\}$$

$$\{\mathrm{F}Q\}_e = [\mathrm{B}200]_e\{\mathrm{QP} - \mathrm{QN}\}_e - \Delta t\Big([\mathrm{B}2J0]_e\{\mathrm{F}J - \mathrm{FV}J\}_e$$

$$+ [\mathrm{A}200]_e\{\mathrm{F}J - \mathrm{FV}J\}_e \hat{\mathbf{n}}_j - \{\mathrm{b(data)}\}_e\Big)_\theta$$

FSNS.9 Depth-Averaged Hyperbolic PDE Verification Problems

Hyperbolic PDE forms for critical flow verifications, *n* = 1, f^v = 0

Hyperbolic divergence form

$$\mathrm{D}M: \ \mathcal{L}(h) = \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(m) = 0, \ \ m = h\bar{u}$$

$$\mathrm{D}P_x: \ \mathcal{L}(m) = \frac{\partial m}{\partial t} + \frac{\partial}{\partial x}\left(\frac{m^2}{h} - \frac{\mathrm{Fr}^2}{2}h^2\right) = 0$$

Primitive variable form

$$\mathrm{D}M: \mathcal{L}(h) = \frac{\partial h}{\partial t} + \bar{u}\frac{\partial h}{\partial x} + h\frac{\partial \bar{u}}{\partial x} = 0$$

$$\mathrm{D}P_x: \mathcal{L}(\bar{u}) = \frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} - \mathrm{Fr}\frac{\partial h}{\partial x} = 0$$

Modified PDEs for TWS^h application require flux vector jacobians

$$f \Rightarrow \begin{Bmatrix} m \\ m^2/h - \frac{1}{2}\mathrm{Fr}^2 h^2 \end{Bmatrix}$$

$$[A] = \begin{bmatrix} 0 & , & 1 \\ -\left(\frac{m}{h}\right)^2 - \mathrm{Fr}^2 h & , & 2\frac{m}{h} \end{bmatrix}$$

$$[AA] = \begin{bmatrix} -\bar{u}^2 - \mathrm{Fr}^2 h & , & 2\bar{u} \\ -2\bar{u}^3 - 2\mathrm{Fr}\bar{u}h & , & 3\bar{u}^2 - \mathrm{Fr}^2 h \end{bmatrix}$$

$$f \Rightarrow \begin{Bmatrix} \bar{u}h \\ \frac{1}{2}\bar{u}^2 - \mathrm{Fr}h \end{Bmatrix}$$

$$[A] = \begin{bmatrix} \bar{u} & , & 0 \\ -\mathrm{Fr} & , & \bar{u} \end{bmatrix}$$

$$[AA] = \begin{bmatrix} \bar{u}\bar{u} & , & 0 \\ -2\mathrm{Fr}\bar{u} & , & \bar{u}\bar{u} \end{bmatrix}$$

FSNS.10 TWS^h + θTS for TS-Modified n = 1 Hyperbolic FSNS

Modified hyperbolic divergence form for primitive form β-term

$$\mathrm{D}M : \mathcal{L}^m(h) = \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(m - \frac{\beta \Delta t}{2} \bar{u}\bar{u} \frac{\partial h}{\partial x} \right) = 0$$

$$\mathrm{D}P_{\mathrm{x}} : \mathcal{L}^m(m) = \frac{\partial m}{\partial t} + \frac{\partial}{\partial x}\left(\bar{u} m - \frac{\mathrm{Fr}^2}{2} h^2 + \beta \Delta t \mathrm{Fr} \bar{u} \frac{\partial h}{\partial x} - \frac{\beta \Delta t}{2} \bar{u}\bar{u} \frac{\partial m}{\partial x} \right) = 0$$

TWS^h + θTS template pseudo-code

$$\{\mathrm{FH}\}_e = (\ )(\ )\{\ \}(1)[\mathrm{A200}]\{\mathrm{H}P - \mathrm{HN}\}$$
$$+ (-\Delta t)(\ )\{\ \}(0)[\mathrm{A210}]\{\mathrm{M}\}_\theta + (\Delta t)(\ )\{\ \}(\ )[\mathrm{ONE}]\{\mathrm{M} \cdot \hat{\mathbf{n}}\}_\theta$$
$$+ (1/2\beta_h \Delta t^2)(\ )\{\overline{\mathrm{U}\mathrm{U}}\}(-1)[\mathrm{A3011}]\{\mathrm{H}\}_\theta$$

$$\{\mathrm{FM}\}_e = (\ )(\ )\{\ \}(1)[\mathrm{A200}]\{\mathrm{M}P - \mathrm{MN}\}$$
$$+ (-\Delta t)(\ )\{\overline{\mathrm{U}}\}(0)[\mathrm{A3010}]\{\mathrm{M}\}_\theta + (\Delta t, \mathrm{Fr}^2/2)(\ )\{\mathrm{H}\}(0)[\mathrm{A3010}]\{\mathrm{H}\}_\theta$$
$$+ (\Delta t)(\ )\{\ \}(\ )[\mathrm{ONE}]\{\overline{\mathrm{U}}\mathrm{M} \cdot \hat{\mathbf{n}}\}_\theta + (-\Delta t, \mathrm{Fr}^2/2)(\ )\{\ \}(\ )[\mathrm{ONE}]\{\mathrm{H}^2\}_\theta$$
$$+ (-\beta_m \mathrm{Fr} \Delta t^2)(\ )\{\overline{\mathrm{U}}\}(-1)[\mathrm{A3011}]\{\mathrm{H}\}_\theta$$
$$+ (1/2\beta_m \Delta t^2)(\ )\{\overline{\mathrm{U}\mathrm{U}}\}(-1)[\mathrm{A3011}]\{\mathrm{M}\}_\theta$$

FSNS.11 Depth-Averaged TWS^h + θTS, Newton Template, n = 1

Newton jacobian

$$[\mathrm{JAC}]_e = \begin{bmatrix} \mathrm{JHH} , & \mathrm{JHM} \\ \mathrm{JMH} , & \mathrm{JMM} \end{bmatrix}_e$$

template pseudo-code for q-Newton

$$[\mathrm{JHH}]_e = (\ )(\ )\{\ \}(1)[\mathrm{A200}][\ ] + (1/2\beta_m \theta \Delta t^2)(\ )\{\overline{\mathrm{UU}}\}(\text{-}1)[\mathrm{A3011}][\ ]$$

$$[\mathrm{JHM}]_e = (-\theta\Delta t)(\ )\{\ \}(0)[\mathrm{A210}][\ ] + (\theta\Delta t)(\ )\{\ \}(\ )[\mathrm{ONE}]\lceil \pm 1 \rfloor$$
$$+ (\theta\beta_h \Delta t^2)(\ )\{\mathrm{H}\}(-1)[\mathrm{A3110}][\mathrm{M/HH}\rfloor$$

$$[\mathrm{JMM}]_e = (\ )(\ )\{\ \}(1)[\mathrm{A200}]\lceil\ \rfloor$$
$$+ (-\theta\Delta t)(\ )\{\overline{\mathrm{U}}\}(0)[\mathrm{A3010}]\lceil\ \rfloor + (-\theta\Delta t)(\ )\{M\}(0)[\mathrm{A3010}]\lceil 1/\mathrm{H}\rfloor$$
$$+ (-\theta\Delta t)(\ )\{\ \}(\ )[\mathrm{ONE}][2M/\mathrm{H}\cdot\hat{\mathbf{n}}\rfloor + (1/2\beta_m\theta\Delta t^2)(\ )\{\overline{\mathrm{UU}}\}(\text{-}1)[\mathrm{A3011}]\lceil\ \rfloor$$

$$[\mathrm{JMH}]_e = (\theta\Delta t)(\ )\{M\}(0)[\mathrm{A3010}]\lceil \mathrm{M/HH}\rfloor + (\theta\Delta t, \mathrm{Fr}^2)(\ )\{\mathrm{H}\}(0)[\mathrm{A3010}]\lceil\ \rfloor$$
$$+ (\theta\Delta t)(\ )\{\ \}(\ )[\mathrm{ONE}][\pm(\mathrm{M/H})^2\rfloor + (-\theta\Delta t, \mathrm{Fr}^2)(\ )\{\ \}(\ )[\mathrm{ONE}]\lceil \mathrm{H}\rfloor$$
$$+ (-\beta_m \mathrm{Fr}\theta\Delta t^2)(\ )\{\overline{\mathrm{U}}\}(-1)[\mathrm{A3011}][\ \rfloor$$

FSNS.12 Verification, Depth-Averaged Flows Over a Bed Profile, $n = 1$

Analytical solution = $f\,(h_{in}, Fr_{in})$

$$\text{sub-critical flow: } \mathrm{Fr}_{in} = 0.3$$

$$\text{BCs: } \bar{u}_{in} = \text{data}$$

$$h_{out} = \text{data}$$

$$\frac{\partial q}{\partial x} \cong 0 \ \text{ elsewhere}$$

$$\text{super-critical flow: } \mathrm{Fr}_{in} = 2.0$$

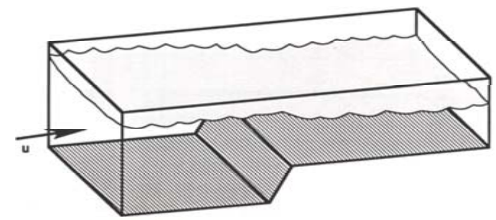
$$\text{BCs: } m_{in}, h_{in} = \text{data}$$

$$\left.\frac{\partial q}{\partial x}\right|_{out} \cong 0$$

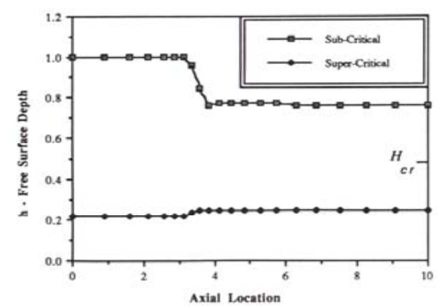
Comparison solutions, TWS^h, $k = 1$ (mks units)

solution	Fr_{in}	ΔFr	Δh	$\Delta \bar{u}$	β_q
analytical TWS^h	0.3 0.3	0.123 0.124	-0.210 -0.215	0.347 0.348	--- (0.3,0.3)
analytical TWS^h	2.0 2.0		0.02 0.02	-0.82 -0.80	--- (0.3,0.3)

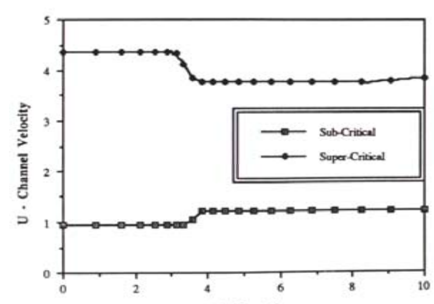
Channel geometry



Flow depth - $h(x)$



Velocity $-\bar{u}(x)$



FSNS.13 Verification, Depth-Averaged Flow Hydraulic Jump, *n* = 1

Supercritical onset flow may create a hydraulic jump

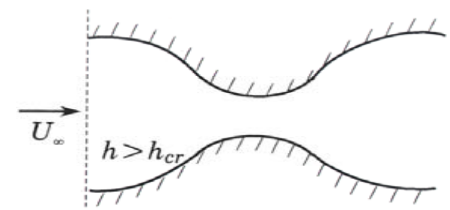
Solutions dependent on $A(x)$, C_z

$A(x) \cong$ de Laval nozzle

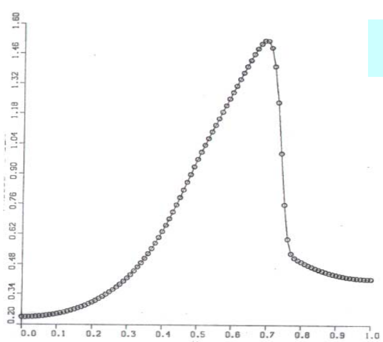
$C_z \equiv$ Chezy bed friction coefficient

TWS^h β solution, inviscid flow, $C_z \equiv 0$

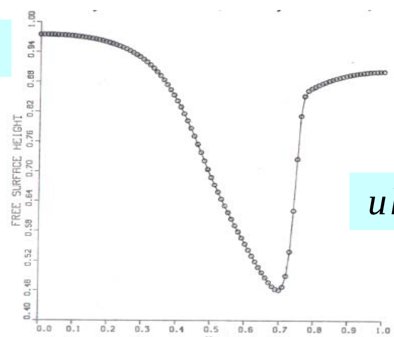
Flow cross-section variation



Fr

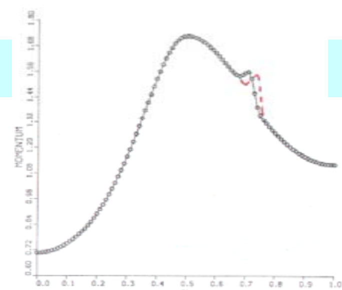


h

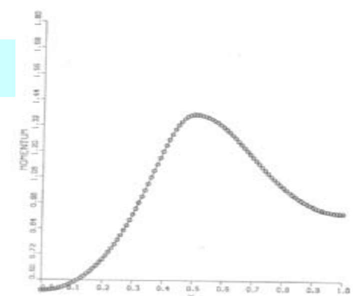


TWS^h β solutions, C_z = 10, 2

uh



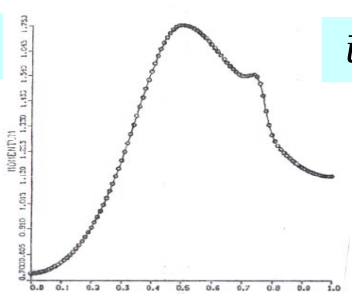
uh



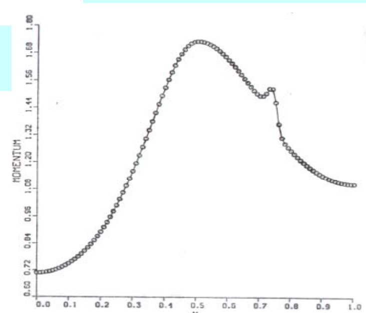
β = {0.2, 0.05}

β_q = {0.1, 0.05}

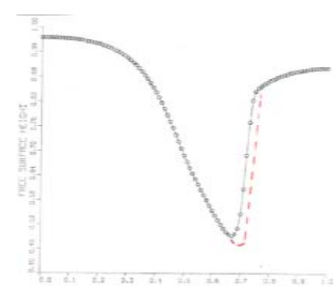
$\bar{u}h$



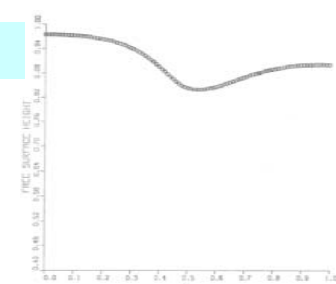
$\bar{u}h$



h



h

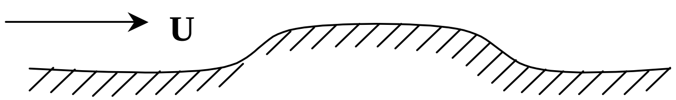


FSNS.14 Verification, TWS^h+θTS, Free-Surface Flow On Bed Profile

Free surface flow PPNS validation, laminar flow over bed profile

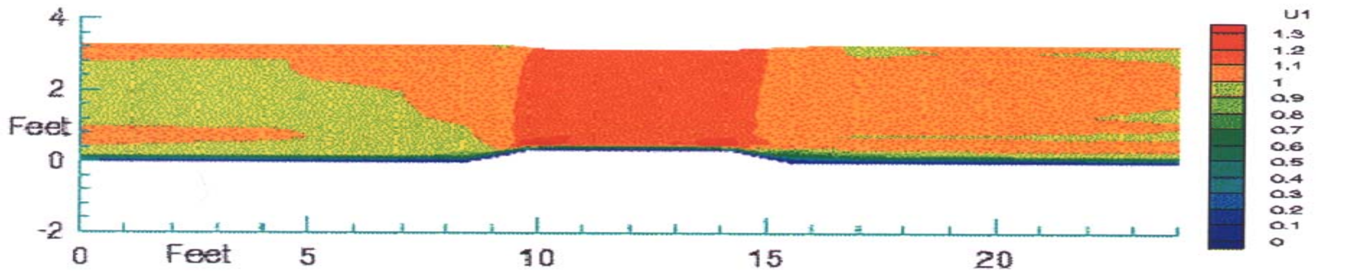
$$\begin{aligned} \text{inflow BCs}: &\ u_1(y) = u_{in},\ h = h_{in},\ Fr_{in} = 0.478 \\ \text{outflow BCs}: &\ \hat{\mathbf{n}} \cdot \nabla q = 0 \\ \text{bed BCs}: &\ u = 0 = v = w \\ \text{free-surface}: &\ h = h\ (\text{depth-averaged solution}),\ \mathbf{u} \cdot \hat{\mathbf{n}} = 0 \end{aligned}$$

bed profile

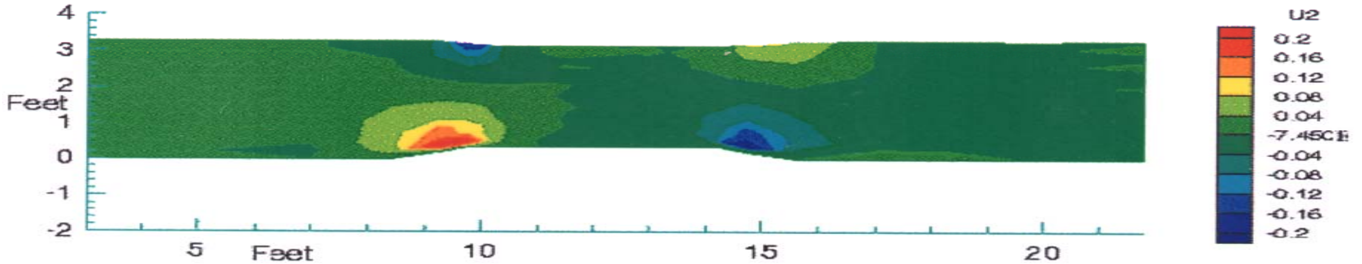


TWS^h + θTS solution, steady-state, $\beta_q = \{0.1, 0.1, 0, 0\}$

u_1



u_2



FSNS.15 Turbulent Depth-Averaged Free-Surface Flow TWS^h + θTS

Partially-parabolic conservation PDE system given on FSNS.5

$$\text{TWS}^h\ \beta-\text{stability approximation uses } A_j A_k \approx [\bar{u}_i \bar{u}_j]$$

$$\text{hence: } \text{TWS}^h + \theta\text{TS} = \text{S}_e\{\text{WS}\}_e = \{0\}$$

$$\{\text{F}Q\}_e = [\text{B200}]_e\{\text{Q}P - \text{QN}\}_e + \Delta t\{\text{RES}(Q, \bar{\text{u}}, \text{Re}, \text{Re}^t, \text{Fr}, ...)\}_\theta$$

Template pseudo-code essence summary

$$\begin{aligned}\{\text{FH}\}_e = {} & (\ )(\ )\{\ \}(0;1)[\text{B200}]\{\Delta\text{H}\} \\ & + (-\theta\Delta t, \beta_1)(\ )\{\ \}(\text{E}KI;0)[\text{B20}k]\{\text{M}I\}_\theta \\ & + (\theta\Delta t, \beta_h\Delta t)(\ )\{\overline{\text{U}}I\overline{\text{U}}J\}(\text{E}KI, \text{E}LI;-1)[\text{B30}KL]\{\text{H}\}\end{aligned}$$

$$\begin{aligned}\{\text{FM}I\}_e = {} & (\ )(\ )\{\ \}(0;1)[\text{B200}]\{\Delta\text{M}I\} \\ & + (-\Delta t)(\ )\{\overline{\text{U}}J\}(\text{E}KJ;0)[\text{B30}K0]\{\text{M}I\}_\theta \\ & + (\Delta t, \text{Fr}^2)(\ )\{\text{H}\}(\text{E}KI;0)[\text{B30}K0]\{\text{H}\}_\theta \\ & + \{\text{b}(\text{Re}^t, \overline{\text{U}}, \text{b}, \rho, \text{Co}, \tau_{\text{is}}, \tau_{\text{ib}}, \beta_m)\}_\theta\end{aligned}$$

FSNS.16 Application: TWS^h + θTS, Depth-Averaged Channel Flow

Channel specifications

$$\text{length/width/depth} = 40/3/8 \pm 4\text{m}$$

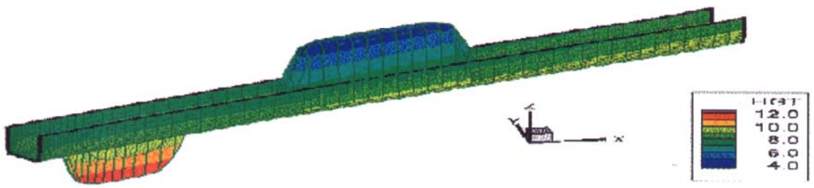
$$\Delta h\big|_{IC} = 0.3\text{m on L}$$

$$\text{BCs}: \ \bar{u}_{1in} = 2.5\text{m/s, slug profile}, \bar{u}_2 = 0, \ h \ \text{floats}$$

$$\text{TKE} \Rightarrow \text{log-law on } 250 \le y^+ \le 600$$

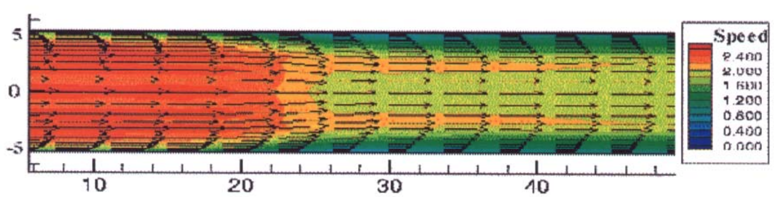
$$\text{data}: \ \beta_q = 0.1(0, 1, 1, 1, 1), \text{Fr} = 0.1, \text{C}_z = 10, \theta = 0.5$$

Geometry perspective

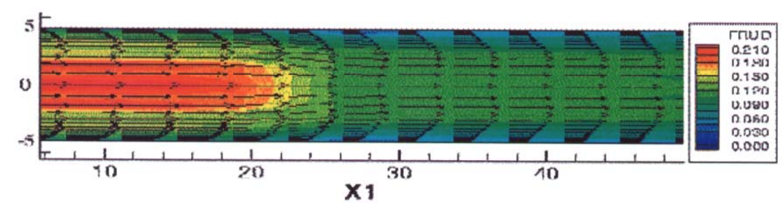


TWS^h steady turbulent flow solution:

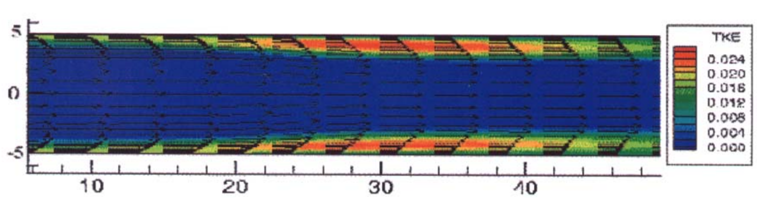
$\bar{U}_1$



Fr



k



FSNS.17 Depth-Averaged TWS^h + θTS, unsteady Tidal Flow Simulation

Tidal flow simulation about cartesian and tear-drop surface penetration islands

$$\text{domain span}: \pm 10^4 \text{ meters}$$

$$\text{tidal cycle BCs}: h(t) = \text{data on } \partial\Omega_{out}$$

$$u_1(t) = \text{data on } \partial\Omega_{in}$$

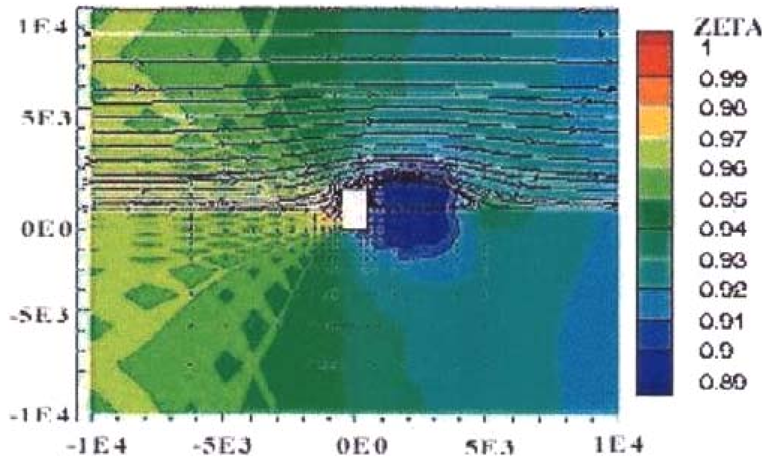
$$\text{Reynolds Nos}: \text{Re} = 10^5/\text{m}, \ \text{Re}^t = 0$$

$$\text{Froude}: -0.2 \le \text{Fr} \le 0.2$$

$$\text{Courant No}: \text{C} \le 84$$

$$\text{Tidal elevation}: \Delta\zeta \Rightarrow \pm 0.1m$$

$$\text{TWS beta}: \beta_q = 0.2\{0, 1, 1\}$$



Mesh resolution study, cartesian island

Ω^h	$\Delta x(x)$	$\Delta x(y)$	$\Delta y(x)$	$\Delta y(y)$	Comments
33×33	40	40	42	36	base
43×55	40	20	42	20	tangential resolution
43×55	3	20	3	20	normal resolution
59×37	3×6	20	3×6	20	uniform normal resolution, half-domain

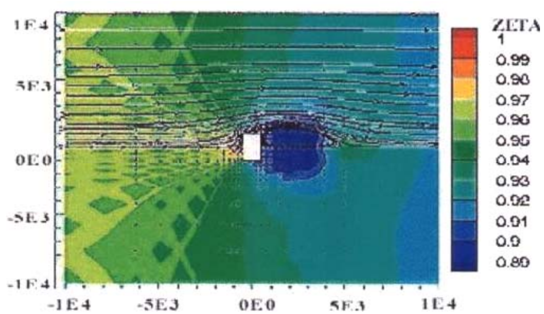
Simulation graphics (aPSE page)

FSNS.18 Mesh Adequacy Assessment Via Color Graphics

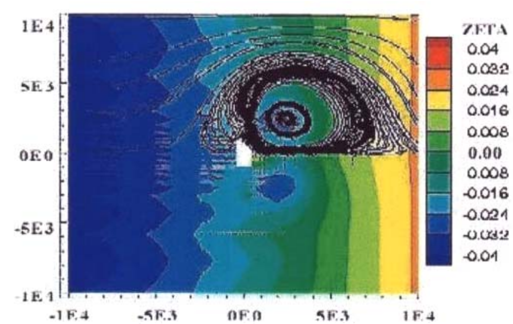
CFD data interpretation employs high performance color graphics

color "diamonds" ⇔ Ω^h resolution inadequate!

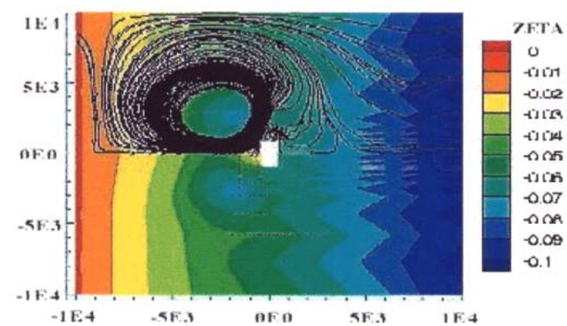
Onset high tide, $\Omega^h = 33^2$



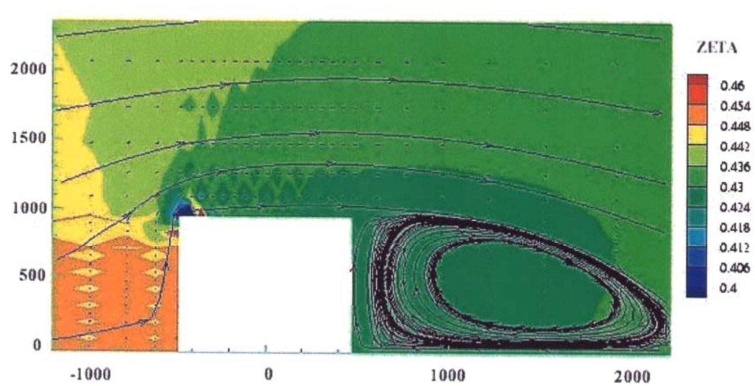
Onset slack tide



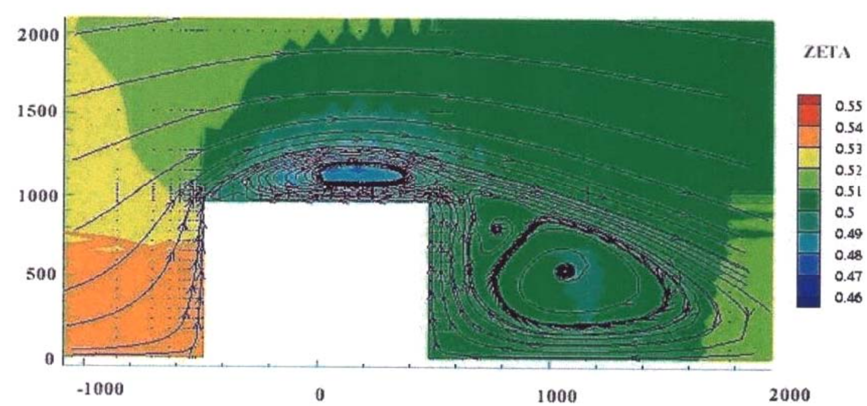
Reverse slack tide



Tangential refinement, $\Omega^h = 43 \times 55$



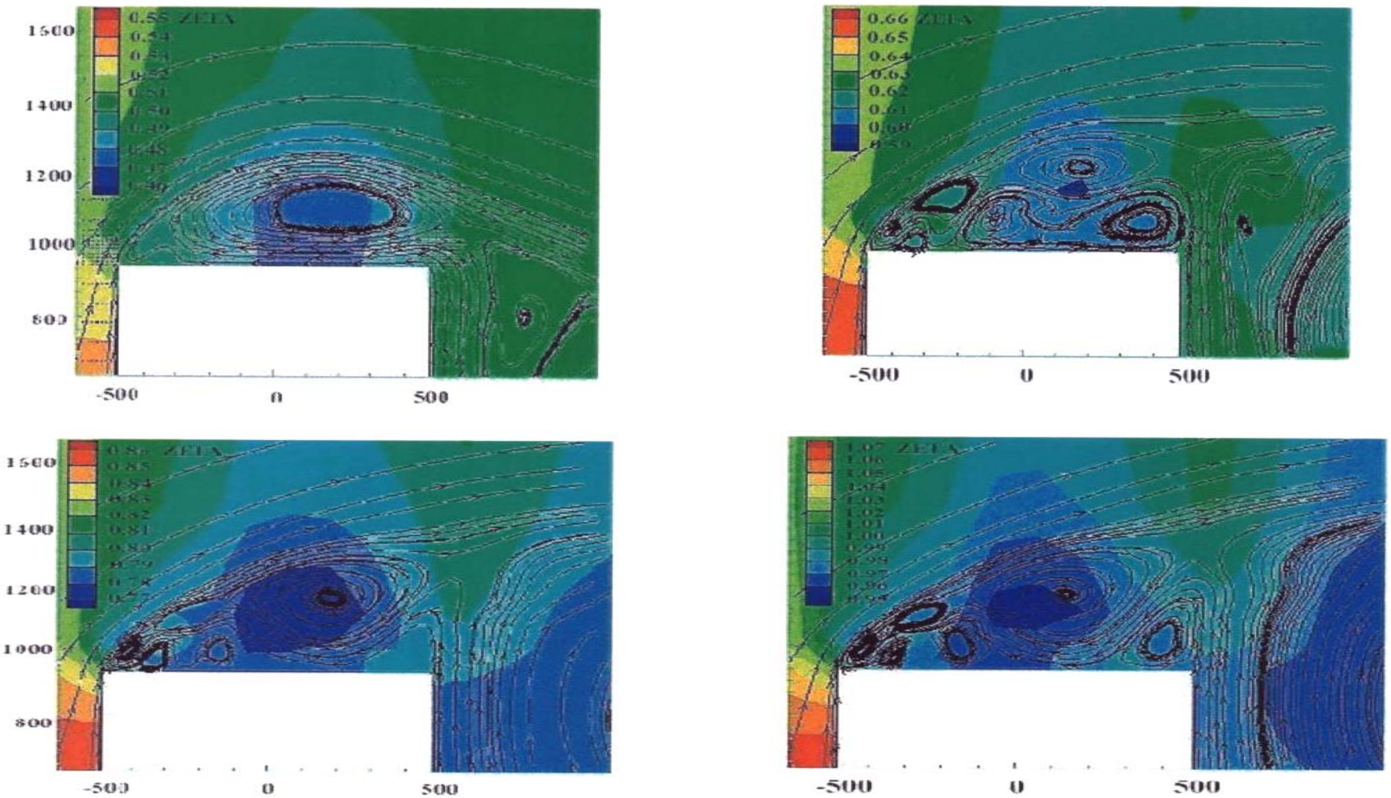
Normal refinement, $\Omega^h = 43 \times 55$



FSNS.19 Mesh Adequacy Assessment Via Color Graphics, Cont’d

Half-cartesian island, “double” mesh refinement, $\Omega^h = 59 \times 37$

Onset tidal cycle, surface elevation on streaklines, t = 10,900s, 13,400s, 17,500s, 21,600s



Tear-drop island tidal cycle (aPSE page)

FSNS.20 Depth-Averaged TWS^h + θTS, Unsteady Tidal Simulation

Tidal flow over sea bed excavation, onset flow at 45°

domain span : ± 3000 meters

excavation depth : 20m

sidewall angle : 45°

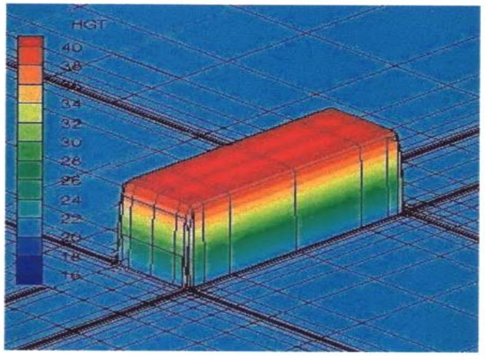
TWS^h beta : $\beta_q = 0.2\{1, 1, 1\}$

tidal variation : $\Delta\zeta \Rightarrow \pm 0.02$m

Froude : $-0.11 \le \text{Fr} \le 0.11$

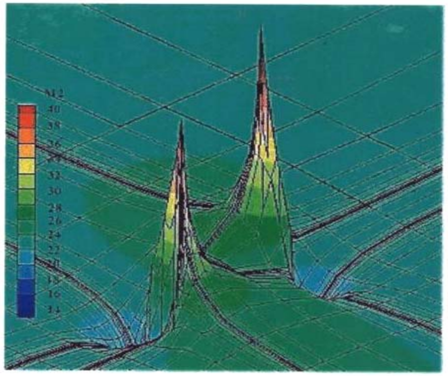
Courant No : $C \le 80$

Bed topology on $h(0)$

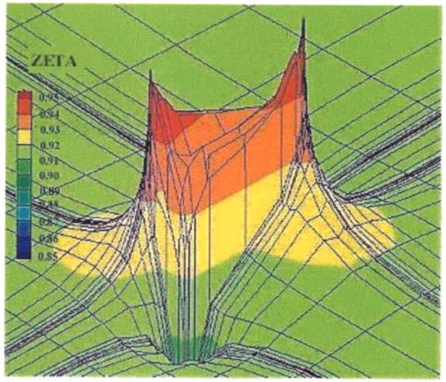


Extremum variations, $\Omega^h = 33^2$ non-uniform

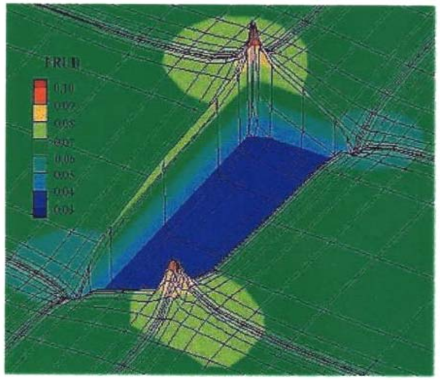
Vertical flux, $m_2 = u_2 h$



Free-surface, ζ



Froude No.

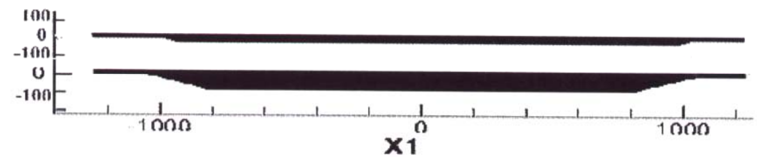


FSNS.21 Free-Surface TWS^h + θTS, Bed Excavation Flowfields

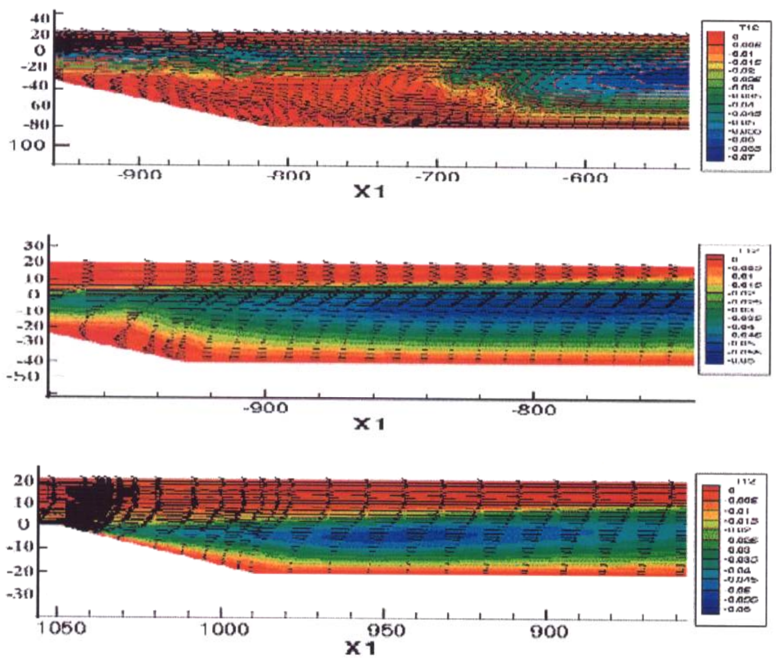
Non-tidal turbulent free-surface flow over a bed excavation, PPNS algorithm

excavation depths: 6.5, 13, 26 meters
onset steady flow: Fr = 0.1, h_{in} = 20 m
domain span: 100 × 1100 m

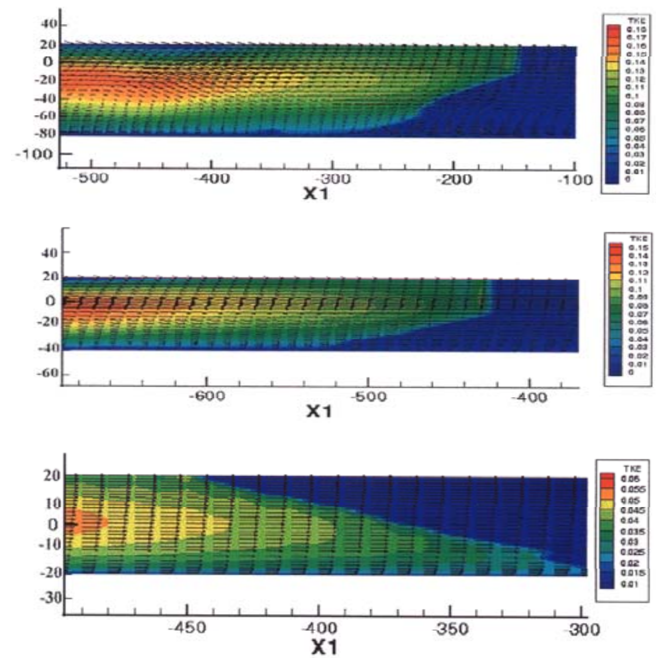
Bed excavation profiles



Nearfield velocity on Reynolds τ_{12}



Farfield velocity on TKE



Excavation unsteady flowfield (aPSE page)

FSNS.22 Summary: Free-Surface Hydrostatic Flows

Tidal, unsteady free-surface flows characterized by Re-averaged INS

$$\text{hydrostatic: } \mathrm{DP}_z \Rightarrow \partial p / \partial z + \rho g = 0$$
$$\text{depth-averaged: } \mathrm{D}M \Rightarrow \partial h / \partial t + \nabla \cdot \mathbf{m} = 0$$

TWS^h + θTS algorithms developed for both forms

hydrostatic: add higher order $\mathrm{D}\mathbf{P}_z$ and employ PPNS theory

depth-averaged: partially-parabolic PDE requires BC attention

$\mathrm{Fr}_{in} < 1$: $\mathbf{m} \cdot \hat{\mathbf{n}}\big|_{in}$ cannot be specified

$\mathrm{Fr}_{in} > 1$: $h, \mathbf{m}\big|_{in}$ required data

Computational experiments confirm TWS^h + θTS robustness

time-accurate unsteady capabilities
TWS^h β-stability is phase selective
verification and benchmark problems illustrated