INS.1 Incompressible Navier-Stokes, CFD-NHT

Fluid-thermal-structural system design uses "CFD with NHT"

for Reynolds-averaged, turbulent, unsteady incompressible flow

DM: $\nabla\cdot\mathbf{u} = 0$

D$\mathbf{P}$: $$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \mathrm{Eu}\nabla p - \nabla\cdot\left(\mathrm{Re}^{-1} + \nu^t\right)\nabla\mathbf{u} + \frac{\mathrm{Gr}}{\mathrm{Re}^2}\Theta\hat{\mathbf{g}} = \mathbf{0}$$

DE: $$\frac{\partial\Theta}{\partial t} + \mathbf{u}\cdot\nabla\Theta - \nabla\cdot\left(\mathrm{Pe}^{-1} + \mathrm{Pr}^{-1}\nu^t\right)\nabla\Theta - s_\Theta = \mathbf{0}$$

D$E(k)$: $$\frac{\partial k}{\partial t} + \mathbf{u}\cdot\nabla k - \nabla\cdot\left(\mathrm{Pe}^{-1} + \nu^t/\mathrm{Pr}^t\right)\nabla k + \mathbf{T}\nabla\mathbf{u} - \varepsilon = \mathbf{0}$$

D$E(\varepsilon)$: $$\frac{\partial\varepsilon}{\partial t} + \mathbf{u}\cdot\nabla\varepsilon - \nabla\cdot\left(\mathrm{C}_\varepsilon\nu^t/\mathrm{Pr}^t\right)\nabla\varepsilon + \mathrm{C}_\varepsilon^1\mathbf{T}\frac{\varepsilon}{k}\nabla\mathbf{u} - \mathrm{C}_\varepsilon^2\varepsilon^2/k = \mathbf{0}$$

where:

non-D parameters are familiar
T is Reynolds stress tensor
DM acts as differential constraint
$\Rightarrow$ no differential expression for pressure p

INS.2 Incompressible N-S, Well-Posedness

Consider unsteady isothermal laminar NS

D*M*: $\nabla \cdot \mathbf{u} = 0$

D**P**: $\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \mathrm{Eu} \nabla p - \mathrm{Re}^{-1} \nabla^2 \mathbf{u} = 0$

$\Rightarrow$ 4 PDEs on $\mathbf{u}$, none on p !

D*M* **acts as a** ***differential constraint*** **on solutions to** D**P**

CFD approaches to enforce constraint include

vector field theory: D*M* guarantees $\mathbf{u} = \nabla \times \mathbf{\Psi}$

$\nabla \times$D**P** eliminates pressure appearance

pressure projection: $\left\| \nabla^h \cdot \mathbf{u}^h \right\| \Rightarrow \varepsilon > 0$, iteratively

pseudo-compressibility: D*M* $\Rightarrow \mathrm{L}(p) = \beta^{-1} p_t + \nabla \cdot \mathbf{u} = 0$

free-surface hydrodynamics: D*M* $\Rightarrow \mathrm{L}(h) = h_t + \nabla \cdot h\mathbf{u} = 0$

INS.3 Taxonomy of INS Algorithms for D*M*

Continuity Enforcement	Method	Origins	Issues/ Detractions
Exact with vorticity	Vorticity stramfunction	Fromm (1963, FD) Baker (1973, FE)	Practical for 2D only, vorticity BC at walls
	Vorticity vector potential	Aziz and Helhums (1967, FD)	BCs for vector potential, vorticity BC
	Vorticity/vector scalar potentials	Aregbesola and Burley (1977, FD)	6 DOF/node in 3D vorticity BC
	Vorticity velocity	Fasel (1976, FD) Deams et al. (1979, FD) Wong & Baker (2001, 3D FE)	6 DOF/node in 3D vorticity BC
Exact	a-P Direct (mixed finite elements)	Ladyzhenskaya (1969, FE) Babuska (1973, FE) Brezzi (1974, FE)	Ill-conditioned numerical diffusion
Inexact Algebraic	Penalty	Temam (1965, FE) Zienkiewicz and Godbole (1975, FE)	Ill-conditioned reduced integration
Inexact Initial value	Pseudo-compressibility	Chorin (1967, FD) Carter (1990, FE)	Steady-state only numerical diffusion

INS.4 Taxonomy of INS Algorithms for D*M*

Continuity Enforcement	Method	Origins	Issues/ Detractions
Inexact Boundary Value Problem	MAC/SMAC	Harlow and Welch (1965, FD)	Staggered mesh velocity BC
	Projection	Chorine (1968, FD) Temam (1969, FD)	Staggered or non-staggered meshes BC implementation
	SIMPLE, SIMPLER SIMPLEC, SIMPLEST	Patankar and Spalding (1972, FD)	Staggered mesh, slow convergence, BC implementation
	Velocity correction	Schneider et al. (19xx, FE)	Equal-order finite element, explicit with lumped mass matrix
	PISO	Issa (1985, FD)	2-step predictor/corrector, staggered mesh, BC, pressure
	Operator splitting	Glowinski (1985, FE)	Decouples non-linearity from incompressibility
	Continuity constraint	Williams and Baker (1999, FE)	Implicit, equal-order, time accurate, finite element Galerkin weak statement