PNS.1 Aerodynamics, Constitutive Closure Models

Conservation principles, compressible flow

$$\mathrm{D}M : \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0$$

$$\mathrm{D}\mathbf{P} : \frac{\partial \rho \mathbf{V}}{\partial t} + \nabla \cdot \rho \mathbf{V}\mathbf{V} = \rho \mathbf{g} + \nabla \mathbf{T}$$

$$\mathrm{D}E : \frac{\partial \rho e}{\partial t} + \nabla \cdot (\rho e + p)\mathbf{V} = s - \nabla \cdot \mathbf{q}$$

Constitutive closure models ⇒ Navier-Stokes equations

Stokes viscosity

$$\mathbf{T} \equiv -p\mathbf{I} + \mu \cdot \nabla \mathbf{V} - \frac{2\lambda}{3}(\nabla \cdot \mathbf{V})\delta$$

Fourier conduction

$$\mathbf{q} = -k \nabla T$$

perfect gas, internal energy

$$p = \rho \mathrm{R} T, \quad e = c_v T, \quad c = \sqrt{\gamma \mathrm{R} T}$$

where: μ, λ, k, R, c_v are fluid properties (data)

PNS.2 Compressible Navier-Stokes, Aerodynamics Simplification

Characteristics of aerodynamic flows

aerodynamic shapes
- flowfield is uni-directional
- farfield is undistributed

large Reynolds number, $Re/L > 10^6$
- viscous effects strictly local

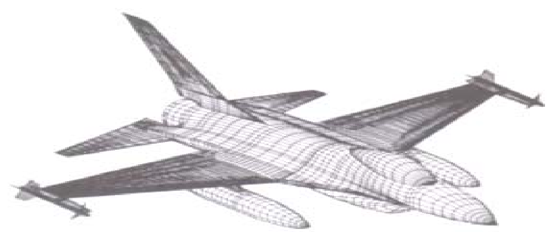
Conservation principle simplifications

farfield described by steady potential flow $\Rightarrow \mathbf{u} = -\nabla\phi$

$$DM: \nabla\cdot\rho(\mathbf{u}) \Rightarrow -\nabla\cdot\rho(\nabla\phi) = 0$$

$$\int D\mathbf{P}\cdot d\mathbf{\Psi} \Rightarrow p(\mathbf{x}) = p_\infty - \frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}$$

Bernoulli pressure



nearfield $DM + D\mathbf{P}$ can be Re-ordered
$\Rightarrow$ boundary layer form of N-S

PNS.3 Aerodynamics – Potential Flow

Inviscid, irrotational steady flow

subsonic – transonic – supersonic: Mach $\equiv \mathrm{M} = \sqrt{U_\infty^2/\gamma R T}$

$$\mathrm{M} < 0.3: \quad \mathrm{D}M = \nabla\cdot\mathbf{u} \Rightarrow -\nabla^2\phi = 0$$

$$\mathrm{M} \approx 1: \quad \mathrm{D}M = \nabla\cdot\rho\mathbf{u} \Rightarrow \left(1-\mathrm{M}_\infty^2\right)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - \mathrm{M}_\infty^2\left[\frac{1+\gamma}{U_\infty}\right]\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2} = 0$$

$$\mathrm{M} > 1: \quad \mathrm{D}M = \nabla\cdot\rho\mathbf{u} \Rightarrow \left(1-\mathrm{M}_x^2\right)\frac{\partial^2\phi}{\partial x^2} + \left(1+\mathrm{M}_y^2\right)\frac{\partial^2\phi}{\partial y^2} - \frac{2}{c^2}\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y} = 0$$

Mach ≤ 0.3

Mach ≈ 1

Mach > 1

PNS.4 Aerodynamics, Weak Interaction Theory

Farfield, subsonic-transonic potential flow assumption

D*M*:

$$L(\phi) = (1 - M_\infty^2)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

$$\ell(\iota) = \hat{\mathbf{n}} \cdot \nabla \phi - U_\infty \hat{\mathbf{i}} \cdot \hat{\mathbf{n}} = 0$$

D*E*:

$$p(\mathbf{x}_\delta) = p_\infty - \rho \nabla \phi \cdot \nabla \phi / 2$$

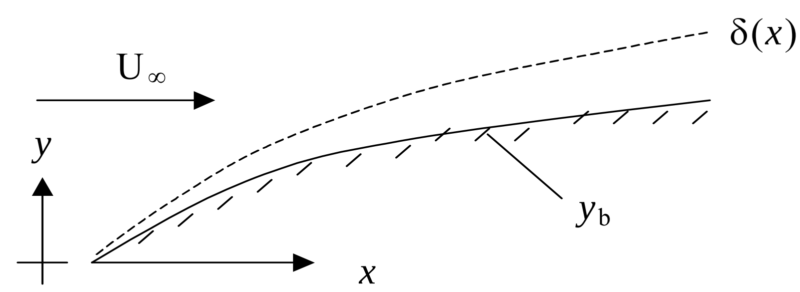
Nearfield, boundary layers wash aerosurfaces

viscous, turbulent effects dominate

D*M*: $\nabla \cdot \mathbf{u} = 0$

D**P**: $\mathbf{u} \cdot \nabla \mathbf{u} = \frac{-1}{\rho_0} \nabla p + \nabla \mathbf{T}$

in the region $y_b(x) \leq y(x) \leq \delta(x)$



$\delta(x) \equiv$ boundary layer thickness

$\mathbf{T}$ = viscous + turbulent effects

PNS.5 Aerodynamics, Boundary Layer Flow

Reynolds ordering of Navier-Stokes, subsonic, $n = 2$

known scales: U_∞, L, $\delta(x)$

non-D ordering: $u/U_\infty \approx O(1)$, $x/L \approx O(1)$, $\delta/L << O(1)$

D*M*:

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow O(1/1) + O(v/\delta) = 0, \quad \text{hence}, \; v/U_\infty \approx O(\delta)$$

D**P**:

$$\mathbf{u} \cdot \nabla \mathbf{u} + \mathrm{Eu} \nabla p - \mathrm{Re}^{-1} \nabla^2 \mathbf{u} = 0$$

$$\hat{\mathbf{i}} \Rightarrow O(1 \cdot 1/1) + O(\delta \cdot 1/\delta) + \mathrm{Eu}\; O(p/1) - \mathrm{Re}^{-1} O(1/1 \cdot 1 + 1/\delta \cdot \delta) = 0$$

$$\text{keeping } \; O(1) \Rightarrow \mathrm{Eu}\; \partial p/\partial x \Rightarrow O(1), \;\; \mathrm{Re}^{-1} \Rightarrow O(\delta^2), \; \frac{\partial^2 u}{\partial x^2} \Rightarrow O(\delta^2)$$

$$\hat{\mathbf{j}} \Rightarrow O(1 \cdot \delta/1) + O(\delta \cdot \delta/\delta) + \mathrm{Eu}\; O(p/\delta) - \mathrm{Re}^{-1} O(\delta/1 \cdot 1 + \delta/\delta \cdot \delta) = 0$$

$$\text{everything } \; O(\delta) \Rightarrow \text{hence } \; \mathrm{Eu}\; \partial p/\partial y = O(\delta), \text{ then } \; p(x,y) \Rightarrow p(x)$$

recall:

$$\mathrm{Re} = \rho_\infty U_\infty L / \mu_\infty$$

$$\mathrm{Eu} = p_\infty / \rho_\infty U_\infty^2$$

PNS.6 Parabolic Navier-Stokes, Boundary Layer Form

Summary, Reynolds ordering of N-S, n = 2, steady subsonic BL

DP_y:	pressure through BL is constant $\Rightarrow P(x)$ from potential farfield DM
DP_x :	$\partial^2 u/\partial x^2$ is $O(\delta^2)$, hence negligible, Re = $O(\delta^{-2}) >> 1$ $\Rightarrow$ parabolic PDE on $x \ge x_0$, $0 \le y \le \delta(x)$
DM:	$\partial v/\partial y = - \partial u/\partial x$, hence initial value on $0 < y \le \delta(x)$

Laminar - thermal subsonic BL non-D conservation form

$$\mathrm{L}(u) = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \mathrm{Eu}\,\frac{\mathrm{d}P_I}{\mathrm{d}x} - \frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial y^2} + \frac{\mathrm{Gr}}{\mathrm{Re}^2}\Theta\hat{\mathbf{g}}\cdot\hat{\mathbf{i}} = 0$$

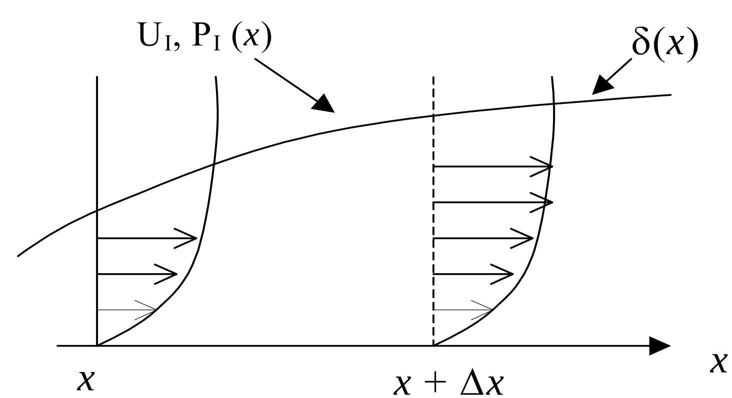
$$\mathrm{L}(\Theta) = u\frac{\partial \Theta}{\partial x} + v\frac{\partial \Theta}{\partial y} - \frac{\mathrm{Ec}}{\mathrm{Re}}\left(\frac{\partial u}{\partial y}\right)^2 - \frac{1}{\mathrm{Pe}}\frac{\partial^2 \Theta}{\partial y^2} = 0$$

$$\mathrm{L}(v) = \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} = 0$$

BCs:

$$u(x, y=0) = 0 = v(x, y=0)$$

$$\left.\frac{\partial u}{\partial y}\right|_{x, y/\delta > 1} = 0, \quad \Theta(x,0) = \Theta_{\mathrm{wall}}, \quad \frac{\partial \Theta}{\partial y}$$



PNS.7 $\text{GWS}^h + \theta\text{TS}$ for Laminar-Thermal BL

$\text{L}(u,\Theta)$ are each of parabolic PDE + BC + IC form

$$\text{L}(q) = u\frac{\partial q}{\partial x} - \frac{1}{\text{Pa}}\frac{\partial^2 q}{\partial y^2} - s(q) = 0$$

$$\text{Approximation}: \ q(x,y) \approx q^N(x,y) \equiv \sum_{\alpha}^{N} \Psi_\alpha(y)\, Q_\alpha(x)$$

$$\text{GWS}^N \Rightarrow [\text{MASS}(u^N)]\{Q\}' + \{\text{RES}\} = \{0\}$$

$$\text{GWS}^N + \theta\text{TS} \Rightarrow \{\text{FQ}\} = [\text{MASS}(u^N)]\{Q - \text{QN}\} + \Delta x\{\text{RES}(Q)\}_{\theta=0.5} = \{0\}$$

$$\text{GWS}^h + \theta\text{TS} \Rightarrow \text{S}_e\{\text{FQ}\}_e \equiv \{0\}$$

$$\{\text{FQ}\}_e = \int_{\Omega_e}\Bigg[\{\overline{\text{U1}}\}^T\{N\}\{N\}\{N\}^T \text{dy}\{\Delta Q\}_e + (\Delta x/2)\{\text{U2}\}_e^T\{N\}\{N\}\frac{\text{d}\{N\}^T}{\text{dy}}\{Q\}_e + \frac{\Delta x}{2\text{Pa}}\frac{\text{d}\{N\}}{\text{dy}}\frac{\text{d}\{N\}^T}{\text{dy}}\{Q\}_e - (\Delta x/2)\{N\}s_e(q)\Bigg]_\theta \text{dy} + \text{BCs}$$

$\text{GWS}^h + \theta\text{TS}$ template pseudo-code

$$\begin{aligned}\{\text{FQ}\}_e = &(\ \ )(\ \ )\{\overline{\text{U1}}\}(1)[\text{A3000}]\{Q\text{P} - \text{QN}\}\\
&+ (\Delta x/2)(\ \ )(\text{U2P})(0)[\text{A3001}]\{Q\text{P}\} + (\Delta x/2)(\ \ )\{\text{U2N}\}(0)[\text{A3001}]\{\text{QN}\}\\
&+ (\Delta x/2, \text{Pa}^{-1})(\ \ )\{\ \ \}(-1)[\text{A211}]\{Q\text{P}\} + (\Delta x/2, \text{Pa}^{-1})(\ \ )\{\ \ \}(-1)[\text{A211}]\{\text{QN}\}\\
&+ (\Delta x)(\ \ )\{\ \ \}(1)[\text{A200}]\{\overline{\text{DPDX}}\} + (\Delta x, \text{Gr}/\text{Re}^2\ \hat{\mathbf{g}}.\hat{\mathbf{i}})(\ \ )\{\ \ \}(1)[\text{A200}]\{\overline{T}\}\\
&+ (-\Delta x/2, \text{Ec}/\text{Re})(\ \ )\{\text{U1P}\}(-1)[\text{A3101}]\{\text{U1P}\} + (-\Delta x/2, \text{Ec}/\text{Re})(\ \ )\{\text{U1N}\}(-1)[\text{A3101}]\{\text{U1N}\}\end{aligned}$$

PNS.8 GWS^h + θTS BL {F(Q)} Completion

$\{F(Q)\}_e$ statement completion for DM

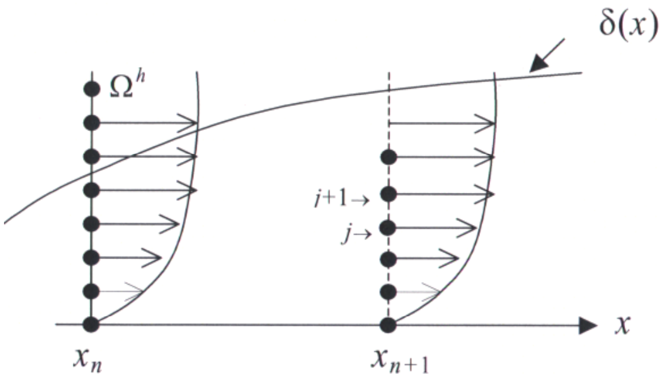
solve: $\mathrm{D}M = \dfrac{\partial v}{\partial y} + \dfrac{\partial u}{\partial x} = 0$ at x_{n+1}

TS:

$$v_{j+1} = v_j + \Delta y \left.\frac{\partial v}{\partial y}\right|_{j+1/2} + O(\Delta y^3)$$

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \Rightarrow \left.\frac{-\,\mathrm{d}\{u\}}{\mathrm{d}x}\right|_{x_{n+1}}$$

$$\frac{\mathrm{d}\{u\}}{\mathrm{d}x} = \mathrm{a}\{\mathrm{U1}\}^{n+1} + \mathrm{b}\{\mathrm{U1}\}^{n} + \mathrm{c}\{\mathrm{U1}\}^{n-1} + O(\Delta x^3)$$



homogenous: $\mathrm{V}_{j+1}^{n+1} - \mathrm{V}_{j}^{n+1} + \ell_e \mathrm{d}\{\mathrm{U1}\}^{n+1} / \mathrm{d}x + O(\Delta x^3) = 0$

hence:

$$\{\mathrm{FU2}\}_e = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} \{\mathrm{U2}\}_e + \frac{1}{2}\ell_e \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \{\mathrm{aU1}^{n+1} + \mathrm{bU1}^{n} + \mathrm{cU1}^{n-1}\}_e$$

GWS^h template: $\{\mathrm{FU2}\}_e = (\quad)(\quad)\{\quad\}(0)[\mathrm{AV1}]\{\mathrm{U2}\} + (1/2)(\quad)\{\quad\}(1)[\mathrm{AV2}]\{\mathrm{U1PR}\}$

matrices:

$$[\mathrm{AV1}] = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}, \quad [\mathrm{AV2}] = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

PNS.9 GWS^h + θTS BL Algorithm Jacobian

Newton Jacobian formed via $\partial\{FQ\}_e / \partial\{Q\}_e$

$$[\mathrm{JAC}]_e = \begin{bmatrix} \mathrm{JUU}, & \mathrm{JUV}, & \mathrm{JUT} \\ \mathrm{JVU}, & \mathrm{JVV}, & 0 \\ \mathrm{JTU}, & \mathrm{JTV}, & \mathrm{JTT} \end{bmatrix}_e$$

aPSE template pseudo-code

$$[\mathrm{JUU}]_e = (\ )(\ )\{\mathrm{U1}\}(1)[\mathrm{A3000}][\ ] + (\Delta x/2)(\ )\{\mathrm{U2}\}(0)[\mathrm{A3001}][\ ] + (\Delta x/2, \mathrm{Re}^{-1})(\ )\{\ \}(-1)[\mathrm{A211}][\ ]$$

$$[\mathrm{JUV}]_e = (\Delta x/2)(\ )\{\mathrm{U1}\}(0)[\mathrm{A3100}][\ ]$$

$$[\mathrm{JUT}]_e = (\Delta x/2, \mathrm{Gr}/\mathrm{Re}^2, \hat{\mathbf{g}}\cdot\hat{\mathbf{i}})(\ )\{\ \}(1)[\mathrm{A200}]\,[\ ]$$

$$[\mathrm{JVU}]_e = (\mathrm{a}, 1/2)(\ )\{\ \}(1)[\mathrm{AV2}][\ ]$$

$$[\mathrm{JVV}]_e = (\ )(\ )\{\ \}(0)[\mathrm{AV1}][\ ]$$

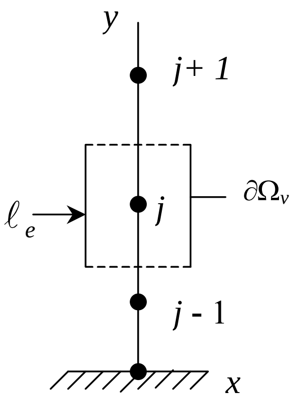
$$[\mathrm{JTU}]_e = (1/2)(\ )\{\mathrm{T}\}(1)[\mathrm{A3000}][\ ] + (\Delta x, \mathrm{Ec/Re})(\ )\{\mathrm{U1}\}(-1)[\mathrm{A3101}][\ ]$$

$$[\mathrm{JTV}]_e = (\Delta x/2)(\ )\{\mathrm{T}\}(0)[\mathrm{A3100}][\ ]$$

$$[\mathrm{JTT}]_e = (\ )(\ )\{\overline{\mathrm{U1}}\}(1)[\mathrm{A3000}][\ ] + (\Delta x/2)(\ )\{\mathrm{U2}\}(0)[\mathrm{A3001}][\ ] + (\Delta x/2, \mathrm{Pe}^{-1})(\ )\{\ \}(-1)[\mathrm{A211}][\ ]$$

PNS.10 $\text{FVS}^h + \theta\text{TS}$ BL Algorithm

FVS^h BL algorithm modifications to GWS^h template



$$\text{FVS}^h = \sum_{\Omega^h} \int_{\Omega_v} \mathrm{L}(u^h)\,\mathrm{d}y = \sum_{\Omega^h} \int_{\Omega_v} \left(\frac{\partial u^2}{\partial x} + P' + \frac{\mathrm{Gr}}{\mathrm{Re}^2} \Theta \hat{\mathbf{g}} \cdot \hat{\mathbf{i}} \right)^h \mathrm{d}y + \oint_{\partial\Omega_v} \left(uv - \frac{1}{\mathrm{Re}} \frac{\partial u}{\partial y} \right)^h \cdot \hat{\mathbf{n}}\,\mathrm{d}\sigma$$

$$\int_{\Omega_v} (\cdot)\mathrm{d}y \Rightarrow 2\ell_e U_j U_j' + \ell_e P' + \ell_e \frac{\mathrm{Gr}}{\mathrm{Re}^2} T_j \hat{\mathbf{g}} \cdot \hat{\mathbf{i}}$$

$$\oint_{\partial\Omega_v} (\cdot) \cdot \hat{\mathbf{n}}\mathrm{d}y = \sum_{\partial\Omega_v = 1}^{2} (\cdot) \cdot \hat{\mathbf{n}} = \left[(VU)_{j+1/2} - (VU)_{j-1/2} \right] - \frac{1}{\ell_e \mathrm{Re}} \left(U_{j+1/2} - U_{j-1/2} \right)$$

$$\equiv \frac{1}{2} V_j (U_{j+1} - U_{j-1}) - \frac{1}{\ell_e \mathrm{Re}} \left(U_{j-1} - 2U_j + U_{j+1} \right) \text{ replacing } j \pm 1/2$$

$\text{FVS}^h + \theta\text{TS}$ template pseudo-code, $\{N_1\}$ equivalent

$$\begin{aligned}
\{\mathrm{FVU}\}_e = {} & (\quad)(U)\{\quad\}(1)[\mathrm{A200F}]\{U\mathrm{P} - \mathrm{UN}\} \\
& + (\Delta x/2)(V)\{\quad\}(0)[\mathrm{A201}]\{U\mathrm{P} + \mathrm{UN}\} \\
& + (\Delta x/2, \mathrm{Re}^{-1})(\quad)\{\quad\}(-1)[\mathrm{A211}]\{U\mathrm{P} + \mathrm{UN}\} \\
& + (\Delta x/2)(\quad)\{\quad\}(1)[\mathrm{A200}]\{\overline{\mathrm{DPDX}}\} \\
& + (\Delta x/2, \mathrm{Gr}/\mathrm{Re}^2, \hat{\mathbf{g}} \cdot \hat{\mathbf{i}})(\quad)\{\quad\}(1)[\mathrm{A200F}]\{\overline{T}\}
\end{aligned}$$

PNS.11 GWS^h, FVS^h + θTS for BL, Accuracy/Convergence

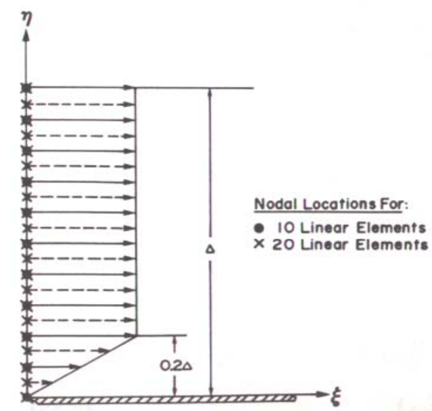
Asymptotic error estimate, GWS^h optimality verification

theory: $\left| e^h(n\Delta x) \right|_E \le C \ell_e^{2k} \left\| \text{data} \right\|_{L2}^2 + C_x \Delta x^3 \left\| U(x_0) \right\|_{H1}^2$

data: $\left\| \text{data} \right\|_{L2}^2 = \int_\Omega (dp/dx)^2 dy = f(x)$

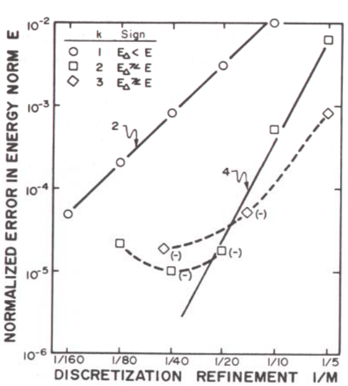
$\left\| U(x_0) \right\|_{H1}^2 = \int_\Omega (U_0)^2 dy + \int_\Omega (dU_0/dy)^2 dy \Rightarrow \text{constant !}$

IC: $\Rightarrow$ must be mesh independent

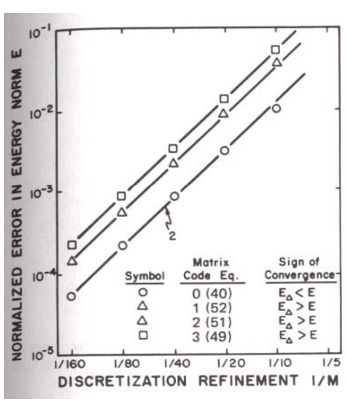


Convergence, optimality (Ch.6, 1983)

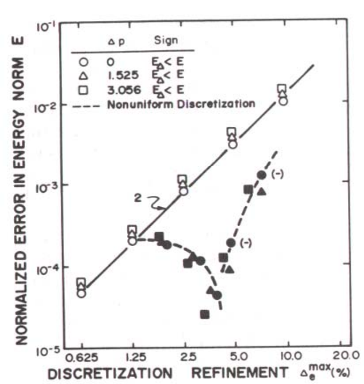
GWS^h, $\{N_k\}$, $1 \le k \le 3$



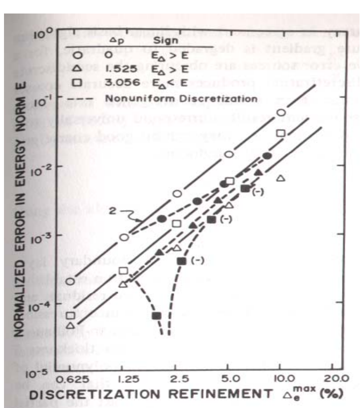
GWS^h optimality, $\{N_1\}$



Regular non-uniform Ω^h refinement, $f(x) > 0$, $f(x) < 0$

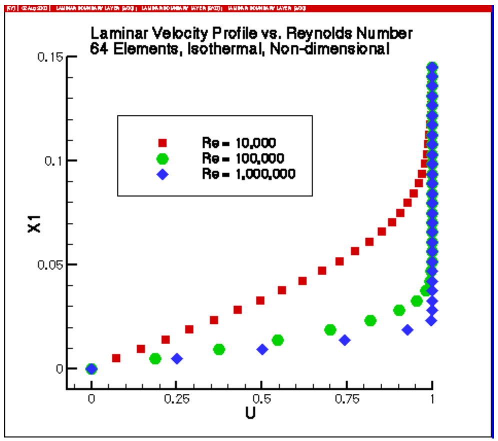


GWS^h ← FVS^h →

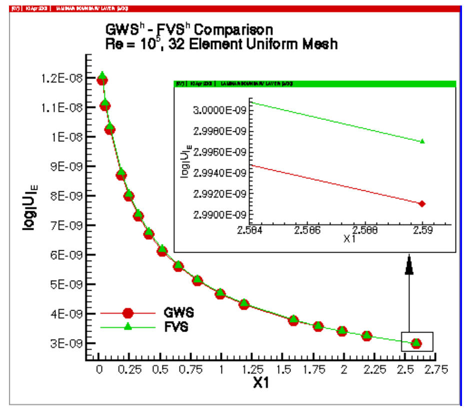


PNS.12 GWS^h, FVS^h + θTS for BL, Accuracy Nuances

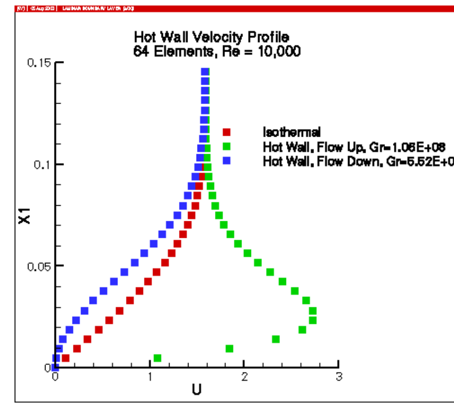
$\{U(n\Delta x)\}$ profiles for Re



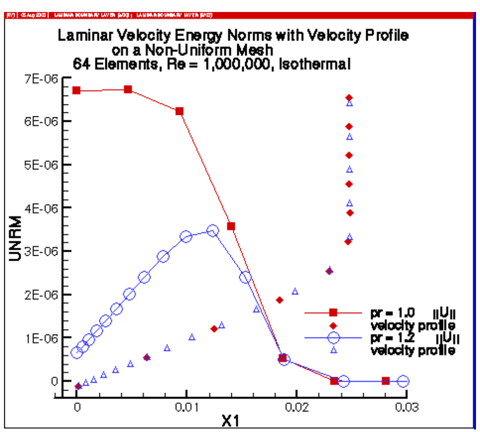
GWS^h optimality



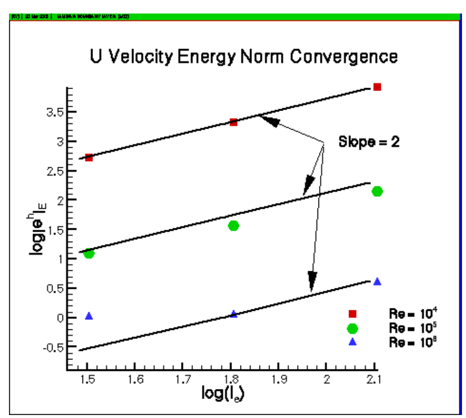
Thermal BLs



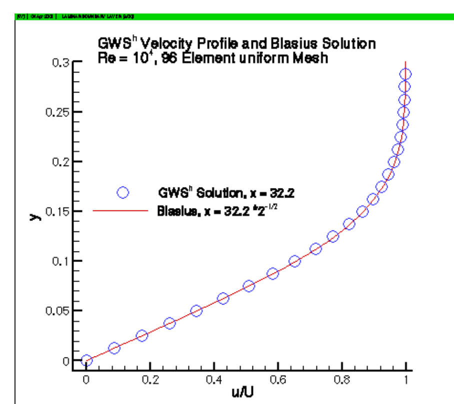
Solution mesh adaptation



GWS^h convergence



GWS^h verification



PNS.13 Boundary Layer Flow, Turbulence

BL form of NS valid only for Re >> 1

aircraft	Mach	U_∞ (m/s)	L (m)	Re	Re/L
commuter	0.3	125	10	3E07	O(E06)
wide body	0.9	250	40	2E08	O(E06)

BL flows will be turbulent (!)

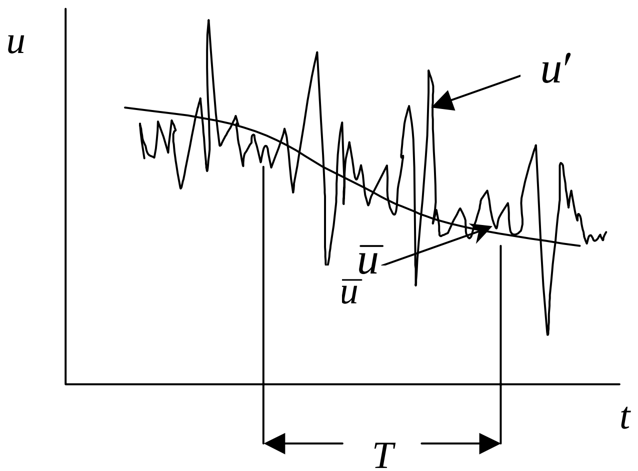
resolution of BL velocity components

$$u(\mathbf{x},t) \equiv \bar{u}(\mathbf{x}) + u'(\mathbf{x},t)$$

time-averaging

$$\bar{u}(\mathbf{x}) \equiv \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} u(\mathbf{x},\tau)\,\mathrm{d}\tau$$

$$\overline{u'} = 0$$



laboratory data

PNS.14 Turbulent Boundary Layer, Reynolds Stress

Time averaging of BL DM and DP

DM: both terms linear, hence $\nabla \cdot \overline{\mathbf{u}} = 0 = \nabla \cdot \mathbf{u}'$

DP$_x$: non-linear convection term generates a new contribution

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \Rightarrow \frac{\partial}{\partial x}(uu) + \frac{\partial}{\partial y}(vv) \quad \text{via D}M$$

$$\overline{uu} \Rightarrow \overline{u}\;\overline{u} + \overline{u'u'}$$
$$\overline{vu} \Rightarrow \overline{v}\;\overline{u} + \overline{v'u'}$$

Reynolds ordering confirms that $O\left(\overline{u'u'}\right) \approx O\left(\overline{v'u'}\right) \approx O(\delta)$

$$\frac{\partial}{\partial x}\left(\overline{u}\;\overline{u} + \overline{u'u'}\right) \Rightarrow O\left(1 \cdot 1/1 + \delta/1\right)$$
$$\frac{\partial}{\partial y}\left(\overline{v}\;\overline{u} + \overline{v'u'}\right) \Rightarrow O\left(\delta \cdot 1/\delta + \delta/\delta\right)$$

hence:

Reynolds normal stress $\overline{u'u'}$ contribution negligible

Reynolds shear stress $\overline{v'u'}$ contribution must be included

PNS.15 Boundary Layer Flow, Turbulence Modeling

Reynolds kinematic shear stress modeled after Stokes

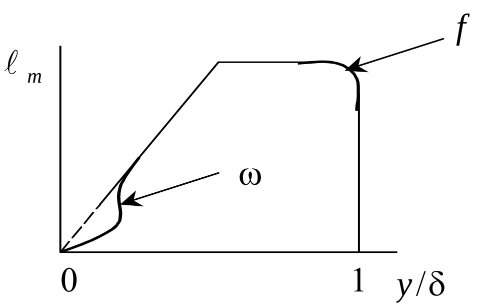
$$\overline{v'u'} \equiv -\upsilon^t \frac{\partial \overline{u}}{\partial y} \quad , \quad \upsilon^t \equiv \text{turbulent "eddy" viscosity, units } (\mu/\rho_\infty = \upsilon) \Rightarrow (\mathrm{L}^2/t)$$

Prandtl mixing length model

$$\upsilon^t \equiv (\omega \ell_m)^2 \left| \frac{\partial \overline{u}}{\partial y} \right| f \Rightarrow (\mathrm{L}^2)(1/t)$$

where: $\ell_m \equiv$ mixing length

$\omega, f =$ near wall, freestream damping



Turbulent kinetic energy-dissipation model

$$\upsilon^t \equiv C_\mu k^2 / \varepsilon \Rightarrow (\mathrm{L}/t)^4 (t^3/\mathrm{L}^2)$$

where:

$$k \equiv \frac{1}{2}\left(\overline{\mathbf{u}' \cdot \mathbf{u}'}\right) = \frac{1}{2}\left(\overline{u'u'} + \overline{v'v'} + \overline{w'w'}\right)$$

$$\varepsilon \equiv \frac{2\upsilon}{3} \left(\overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_i}{\partial x_j}} \right) \delta_{jk}$$

and: $\mathcal{L}(k)$ and $\mathcal{L}(\varepsilon)$ BL forms augment BL DM & DP_x

PNS.16 GWS^h + θTS, Turbulent BL, MLT Closure

Turbulent BL conservation law form, time-averaged $q(x,y)$, MLT

$$\mathrm{D}\mathbf{P}_x: \ \mathrm{L}(\bar{u}) = \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} - \frac{1}{\mathrm{Re}}\frac{\partial}{\partial y}(1+\mathrm{Re}^t)\frac{\partial \bar{u}}{\partial y} + \frac{\mathrm{dP}^I}{\mathrm{d}x} + \frac{\mathrm{Gr}}{\mathrm{Re}^2}\overline{\Theta}\,\hat{\mathbf{g}}\cdot\hat{\mathbf{i}} = 0$$

$$\mathrm{D}\Theta: \ \mathrm{L}(\overline{\Theta}) = \bar{u}\frac{\partial \overline{\Theta}}{\partial x} + \bar{v}\frac{\partial \overline{\Theta}}{\partial y} - \frac{1}{\mathrm{Re}}\frac{\partial}{\partial y}\left(\frac{1}{\mathrm{Pr}} + \frac{\mathrm{Re}^t}{\mathrm{Pr}^t}\right)\frac{\partial \overline{\Theta}}{\partial y} - \frac{\mathrm{Ec}}{\mathrm{Re}}\left(\frac{\partial \bar{u}}{\partial y}\right)^2 \equiv 0$$

$$\mathrm{D}M: \ \mathrm{L}(\bar{v}) = \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{u}}{\partial x} = 0$$

$$\mathrm{D}\mathbf{P}_y: \ \mathrm{L}(\bar{p}) = \frac{\partial}{\partial y}\left(\rho_0 P^I + \overline{v'}\,\overline{v'}\right) = 0$$

$$\text{where}: \ \mathrm{Re}^t \equiv (\upsilon^t/\upsilon)_{\mathrm{dim}} = \text{turbulent Reynolds number}$$

$$\upsilon^t \equiv (\omega \ell_m)^2\left|\frac{\partial \bar{u}}{\partial y}\right| f = \text{MLT eddy viscosity}$$

$$\omega = 1 - \exp(-y/\mathrm{A}) = \text{van Driest damping}, \ \mathrm{A} \approx 25$$

$$\ell_m = \text{Prandtl mixing length} = \begin{cases} \kappa y, & on \ 0 \le y/\delta \le \lambda/\kappa \\ \lambda\delta, & on \ \lambda/\kappa < y/\delta \le 1 \end{cases} \quad \kappa = 0.405, \ \lambda = 0.09$$

$$f = \left[1 + 5.5(y/\delta)^6\right]^{-1} = \text{Klebanoff damping}$$

$$\mathrm{Pr}^t \cong \mathrm{Pr} \text{ for turbulent Prandtl number (usually)}$$

$$\overline{v'}\,\overline{v'} = \text{Reynolds transverse normal stress}$$

PNS.17 GWS^h + θTS Template for Turbulent BL, MLT Closure

Laminar template pseudo-code modifications are modest

$$\text{for } \{\mathrm{F}U\}_e, \{\mathrm{F}T\}_e : (\mathrm{Pa}^{-1}) \cdots [\mathrm{A}211] \Rightarrow (\Delta x/2, \mathrm{Pa}^{-1})(\ )\{\mathrm{RET}\}(-1)[\mathrm{A}3011]\{Q\}$$

$$\text{Reynolds shear sress}: \ \mathrm{L}(\overline{u'v'}) = \overline{u'v'} + \upsilon^t \, \partial u/\partial y = 0$$

$$\mathrm{GWS}^h\left(\mathrm{L}(\overline{u'v'})\right) = \mathrm{S}_e\{\mathrm{WS}\}_e \equiv \{0\}$$

$$\{\mathrm{WS}\}_e = (\ \ )(\ \ )\{\ \ \}(1)[\mathrm{A}200]\{T\mathrm{XY}\}$$

$$+ (\mathrm{Re}^{-1})(\ \ )\{\mathrm{RET}\}(0)[\mathrm{A}3001]\{\mathrm{U}1\}$$

quasi-Newton jacobian, p^{th} iteration

$$\text{solve for } \{\delta\mathrm{U}1, \delta\mathrm{U}2, \delta T\}^{p+1} \ \Rightarrow \{Q\}^{p+1}$$

$$\text{update}: \upsilon^t = (\omega \ell_m)^2 \left|\partial\mathrm{U}1/\partial y\right| f \Rightarrow \{\mathrm{RET}\}^{p+1}$$

$$\text{direct solve}: \text{using these data} \quad \Rightarrow \{T\mathrm{XY}\}^{p+1}$$

compute energy norms for $\{Q\}_{n+1}^{p+1}$ converged

PNS.18 GWS^h + θTS Performance, Turbulent BL, MLT Closure

Accuracy, convergence, *regular* non-uniform Ω^h refinement

theory:

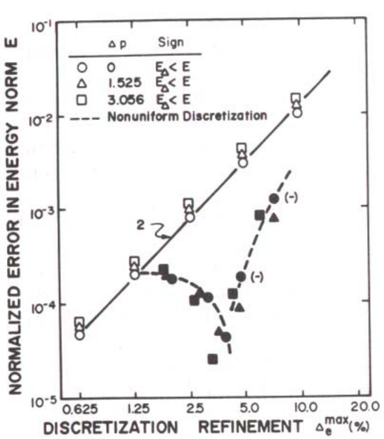
$$\left|e^h(n\Delta x)\right|_E \le C h_e^{2k} \left\|\text{data}\right\|^2_{H^{k-1}} + C_x \Delta x^3 \left\|U_0\right\|^2_{H1}$$

norm:

$$\left|u^h(n\Delta x)\right|_E \equiv \frac{1}{2}\int_\Omega \upsilon^t \left(\frac{\partial u^h}{\partial y}\right)^2 dy = \frac{1}{2}\sum_{\Omega^h}\int_{\Omega_e}(\cdot)dy$$

$$= \frac{1}{2\,\mathrm{Re}}\sum_e^M \{U\}_e^T \{\mathrm{RET}\}_e^T [\mathrm{A3011}]\{U\}_e$$

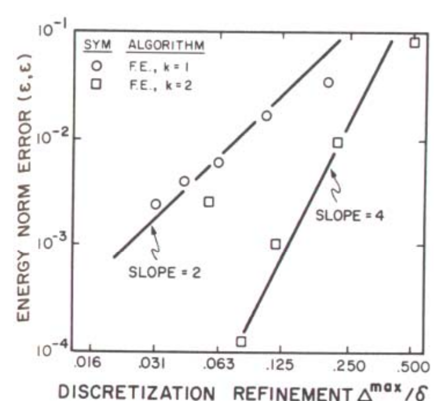
IC, M = 80 laminar



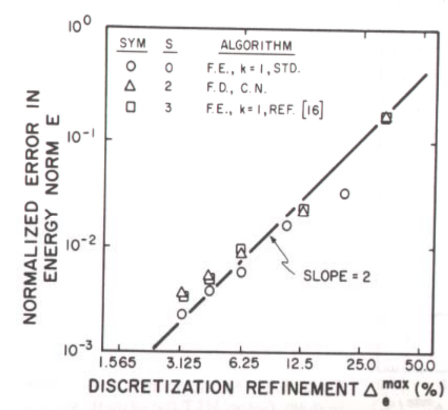
Ω^h progressions

Linear basis, $k=1$		Quadratic basis, $k=2$	
M	ρ	M	ρ
12	1.627	6	2.814
24	1.222	12	1.510
36	1.125	18	1.271
48	1.083	24	1.176
60	1.061	30	1.110

Convergence



Optimality



PNS.19 GWS^h + θTS Validation, Turbulent BL, MLT Closure

Boundary layer theory employs many integral "norms"

displacement thickness : $\delta^*(x) \equiv \int_0^{\delta} \left(1 - u(x,y)/\mathrm{U}^I(x)\right) dy$

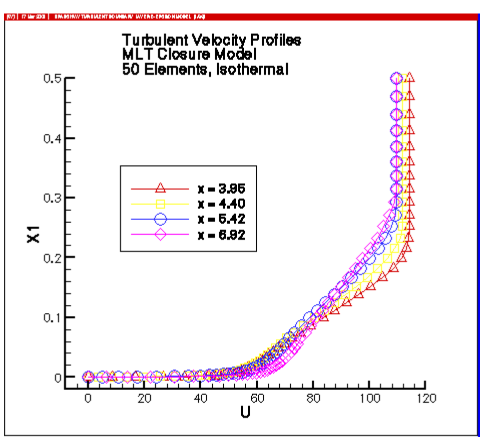
momentum thickness : $\theta(x) \equiv \int_0^{\delta} [u(x,y)/\mathrm{U}^I(x)](1 - u(x,y)/\mathrm{U}^I(x)) dy$

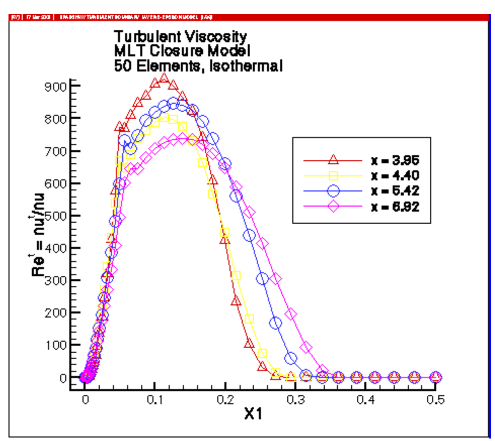
shape factor : $\mathrm{H} \equiv \delta^* / \theta$

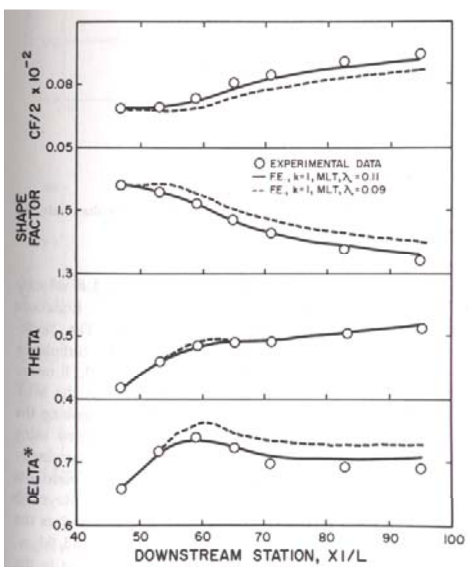
skin friction : $C_f \equiv \tau_w / \rho^I \mathrm{U}^I \mathrm{U}^I / 2$

Ludwig – Tillman : $C_f \equiv 0.246(10)\exp(-0.678\mathrm{H})\,\mathrm{Re}_\theta \exp(-0.268)$

Validation, Bradshaw I-2400 experiment







PNS.20 Turbulent Boundary Layer, TKE Closure

Turbulent kinetic energy-isotropic dissipation closure model

eddy viscosity:

$$\upsilon^t \equiv C_\mu k^2/\varepsilon, \ C_\mu = 0.09$$

$$k \equiv \frac{1}{2}\left(\overline{\mathbf{u}' \cdot \mathbf{u}'}\right) = \frac{1}{2}\left(\overline{u'u'} + \overline{v'v'} + \overline{w'w'}\right) \quad \varepsilon \equiv \frac{2v}{3}\left(\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_j}}\right)\delta_{jk}$$

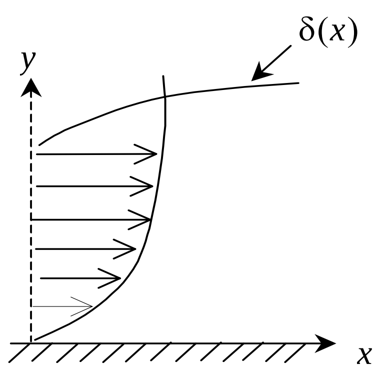
$\mathrm{L}(k, \varepsilon)$ conservation PDEs, non-D BL form

$$\mathrm{L}(k) = u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} - \frac{1}{\mathrm{Pe}}\frac{\partial}{\partial y}\left(1 + \frac{\mathrm{Re}^t}{C_k}\right)\frac{\partial k}{\partial y} - \tau_{12}\frac{\partial u}{\partial y} + \varepsilon = 0$$

$$\mathrm{L}(\varepsilon) = u\frac{\partial \varepsilon}{\partial x} + v\frac{\partial \varepsilon}{\partial y} - \frac{1}{\mathrm{Pe}}\frac{\partial}{\partial y}\left(1 + \frac{\mathrm{Re}^t}{C_\varepsilon}\right)\frac{\partial \varepsilon}{\partial y} - C_\varepsilon^1\frac{\varepsilon}{k}\tau_{12}\frac{\partial u}{\partial y} + C_\varepsilon^2\frac{\varepsilon}{k}\varepsilon = 0$$

$$\mathrm{L}(\tau_{12}) = \tau_{12} + \upsilon^t\frac{\partial u}{\partial y} = \tau_{12} + C_\mu\frac{k^2}{\varepsilon}\frac{\partial u}{\partial y} = 0$$

TKE model adds non-linear parabolic PDE + BCs + IC pair



BCs:

$$k(x, y=0) = 0, \ \varepsilon(x, y=0) \Rightarrow \varepsilon_w < \infty$$

$$\left.\frac{\partial k}{\partial y}, \frac{\partial \varepsilon}{\partial y}\right|_{y \geq \delta(x)} = 0$$

IC:

$$k(x_0, y) = ? = \varepsilon(x_0, y)$$

PNS.21 TKE for Turbulent BL, Near-Wall Corrections

TKE closure model requires near-wall corrections

low Re^t closure model constant modifications (Lam-Bremhorst)

$$\upsilon^t \Rightarrow f_v C_\mu k^2/\varepsilon: \ f_v = \left(1-\exp(-0.0165\,\mathrm{R}_y)\right)^2\left(1+20.5/\mathrm{Re}^t\right)$$
$$C_\varepsilon^1 \Rightarrow f^1 C_\varepsilon^1: \ f_k = (1+0.05\,f_v^{-1})^3$$
$$C_\varepsilon^2 = f^2 C_\varepsilon^2: \ f_\varepsilon = 1-\exp(-\mathrm{Re}^t)^2$$

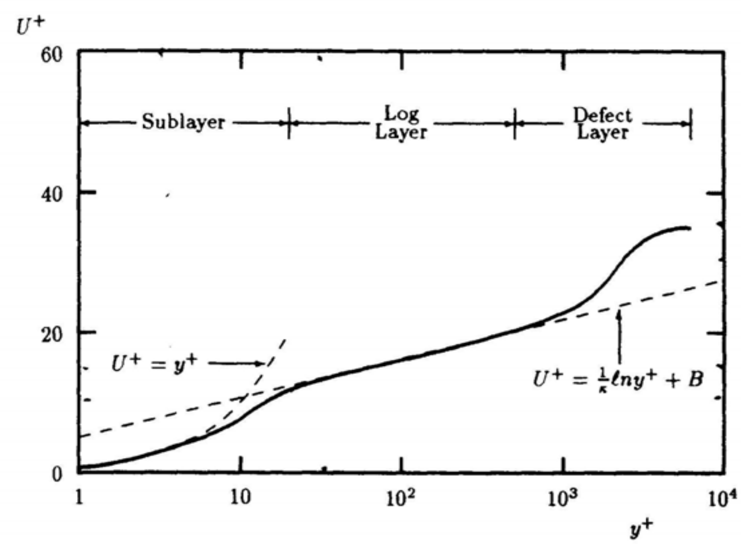
$$\mathrm{Re}^t = \upsilon^t/\upsilon$$
$$\mathrm{R}_y = k^{1/2} y/\upsilon$$

BL similarity TKE variable distributions as *f* (*y*)

$$\mathrm{U}^+ \equiv u/u_\tau = \kappa^{-1}\log(y^+\mathrm{E})+\mathrm{B}$$
$$y^+ \equiv u_\tau y/\upsilon$$

near-wall production = dissipation in $\mathcal{L}$ (*k*)

$$\Rightarrow \quad \upsilon^t = \kappa y u_\tau$$
$$k = u_\tau^2/C_\mu$$
$$\varepsilon = (\kappa y)^{-1}|u_\tau|^3$$
$$\tau_w = \sqrt{C_\mu}\,k$$



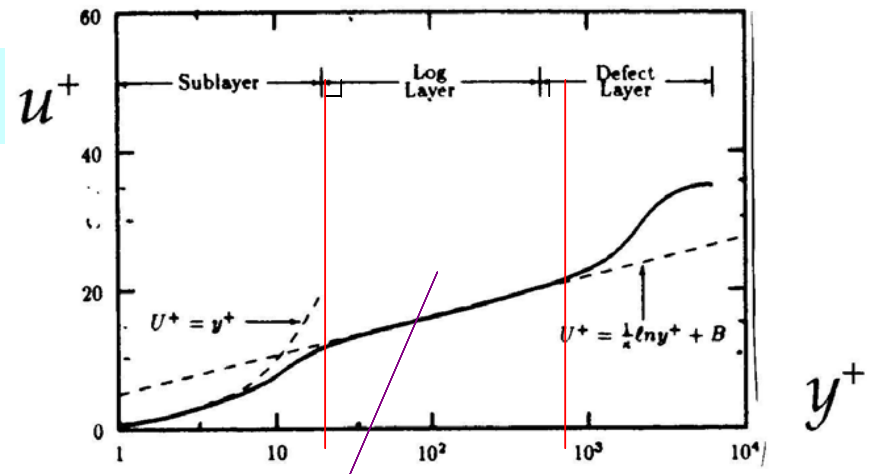
PNS.22 Turbulent Boundary Layer Similarity

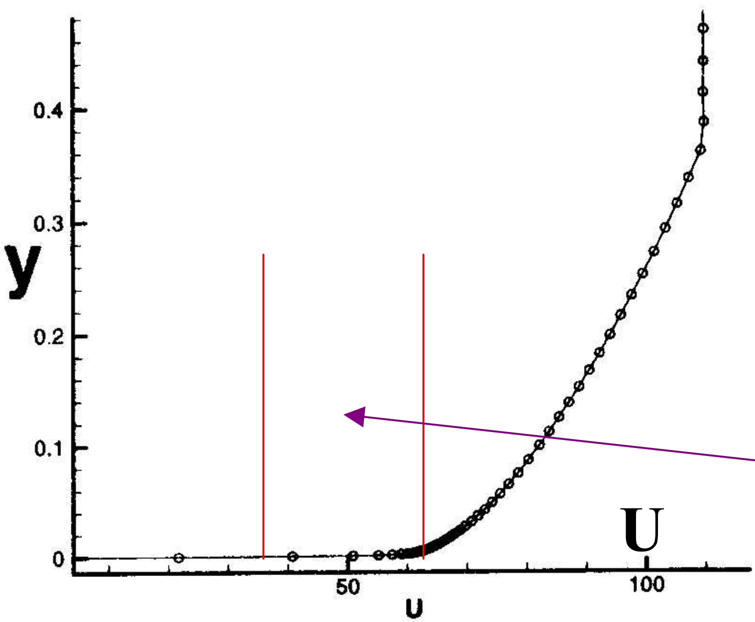
Summary; turbulent BL similarity

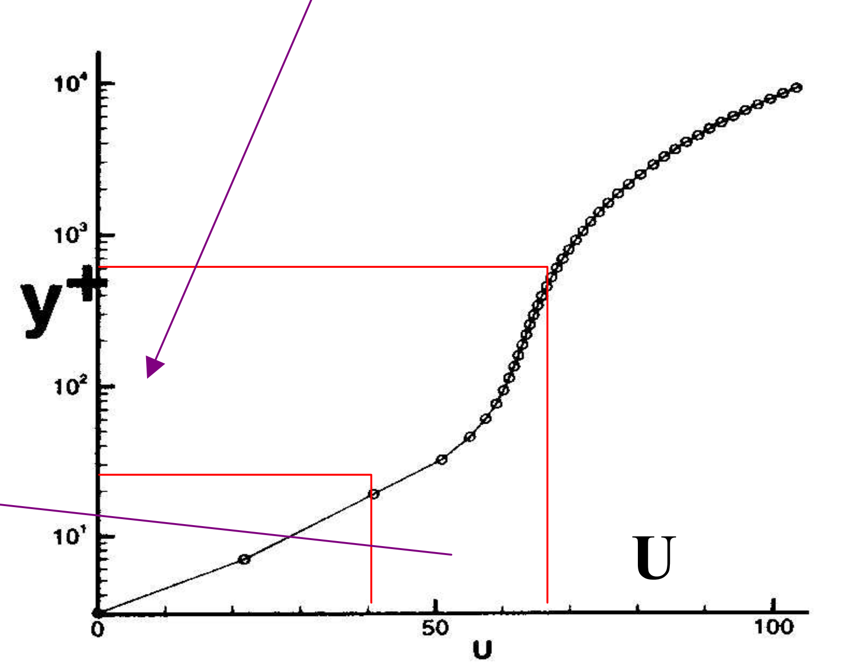
$$u/u_\tau = \frac{1}{\kappa}\,\ell n\left(y^+E\right) + B \equiv u^+$$

$$y^+ = u_\tau y/\upsilon$$

example: IDENT 2400 boundary layer







PNS.23 GWS^h + θTS Template, BL, TKE + Low Re^t

Template pseudo-code modifications ($\{FV\}_e$ unchanged)

for $\{Q\}_e = \begin{Bmatrix} U \\ T \\ K \\ E \end{Bmatrix}_e$:

$$\begin{aligned}\{FQ\}_e &= (\ )(\ )\{\overline{U}\}(1)[A3000]\{QP - QN\} \\ &+ (\Delta x/2)(\ )\{VP, VN\}(0)[A3001]\{QP, QN\} \\ &+ (\Delta x/2, Pa^{-1})(\ )\{RET\}[A3011]\{QP, QN\} + \{b(Q)\}\end{aligned}$$

Source terms $\{b\,(\cdot)\}$ unchanged for $\{Q\} = \{U, T\}^T$, and

$$\begin{aligned}\{b(K)\}_e &= \int_{\Omega_e} \{N\}(\tau_{12}\partial u_e/\partial y)dy + \int_{\Omega_e}\{N\}\varepsilon_e dy \\ &= (\Delta x/2)(\ )\{TXY\}(0)[A3001]\{U\} + (\Delta x/2)(\ )\{\ \}(1)[A200]\{E\} \\ \{b(E)\}_e &= C_\varepsilon^1 \int_{\Omega_e}\{N\}(\tau_{12}(\varepsilon/k)_e \partial u_e/\partial y)dy + C_\varepsilon^2 \int_{\Omega_e}\{N\}(\varepsilon/k)_e \varepsilon_e dy \\ &= (\Delta x/2, CE1)(FE1)\{TXY, E/K\}(0)[A3001]\{E\} \\ &+ (\Delta x/2, CE2)(FE2)\{E/K\}(1)[A3000]\{E\}\end{aligned}$$

Reynolds shear stress template

$$\begin{aligned}\{FTXY\}_e &= (\ )(\ )\{\ \}(1)[A200]\{TXY\} \\ &+ (Re^{-1})(FNU)\{RET\}(0)[A3001]\{U\}\end{aligned}$$

PNS.24 GWS^h + θTS TKE BL, Quasi-Newton Jacobian

Size, deeply embedded non-linearity precludes Newton

quasi-Newton jacobians:

$$\begin{bmatrix} JUU, & JUV, & JUT \\ JVU, & JVV, & 0 \\ JTU, & JTV, & JTT \end{bmatrix}, \quad \begin{bmatrix} JKK, & JKE, & JKT_{xy} \\ JEK, & JEE, & JET_{xy} \\ JT\ K, & JT\ E, & JT\ T_{xy} \end{bmatrix}_e$$

solution sequence:

$\{\delta U, \delta V, \delta T\}^{p+1}$ unchanged from laminar, MLT
update $\{U, V, T\}^{p+1}$
$\{\delta K, \delta E, \delta T_{xy}\}^{p+1}$ uses $\{U, V, T\}^{p+1}$
update $\{K, E, T\}^{p+1}$
index p, return to $\{\delta U, \delta V, \delta T\}^{p+1}$

oscillating convergence:

use {RETN} in $\{FU, FT\}^p$
use {UN, VN} in $\{FK, FE, FT_{xy}\}^p$

templates:

$[JAC]_e$ for $\{FU, FV, FT\}$ are unchanged
$[JAC]_e$ for $\{FK, FE, FT_{xy}\}$ fully utilizes chain rule

PNS.25 GWS^h + θTS TKE Closure Jacobian Coupling

Jacobian coupling for convection terms is unchanged

$$\text{diffusion term}: \quad \mathrm{L}(q) \Rightarrow -\frac{1}{\mathrm{Re}}\frac{\partial}{\partial y}\left(\frac{1}{\mathrm{Pr}} + \frac{\mathrm{Re}^t}{\mathrm{C}_q \mathrm{Pr}^t}\right)\frac{\partial q}{\partial y} \quad , \; q = \{k, \varepsilon\}$$

$$\frac{\partial}{\partial q}(\cdot) = \frac{1}{\mathrm{Re}}\frac{\partial}{\partial y}\left(1 + \frac{\mathrm{Re}^t}{\mathrm{C}_q}\right)\frac{\partial(\cdot)}{\partial y} + \frac{1}{\mathrm{Re}}\frac{\partial}{\partial y}\left(\frac{\partial \tau_{12}^e}{\partial q}\right)\frac{\partial q}{\partial y}$$

$$\frac{\partial \tau_{12}}{\partial q} = \frac{\partial}{\partial q}\left(\mathrm{C}_\mu f_\upsilon k^2/\varepsilon\right) \Rightarrow \begin{cases} 2\mathrm{C}_\mu f_\upsilon k/\varepsilon = 2\tau_{12} k^{-1} \\ -\mathrm{C}_\mu f_\upsilon (k/\varepsilon)^2 = -\tau_{12}\varepsilon^{-1} \end{cases}$$

assuming $\mathrm{Pr}^t \approx \mathrm{Pr}$, hence $\mathrm{RePr} \approx \mathrm{RePr}^t = \mathrm{Pe}$

$$\begin{aligned}
[\mathrm{J}KK]_e = {} & (\Delta x/2, \mathrm{Pe}^{-1})(\;)\{\;\}(-1)[\mathrm{A}211][\;] \\
& + (\Delta x/2, \mathrm{Pe}^{-1}, \mathrm{C}_k^{-1})(\;)\{\mathrm{RET}\}(-1)[\mathrm{A}3011][\;] \\
& + (\Delta x, \mathrm{Pe}^{-1}, \mathrm{C}_k^{-1})(\;)\{K\}(-1)[\mathrm{A}3110][\mathrm{RET}, K^{-1}] \\
& + (\Delta x)(\;)\{U\}(0)[\mathrm{A}3100][T\mathrm{XY}, K^{-1}]
\end{aligned}$$

PNS.26 GWS^h + θTS TKE Closure Jacobian Coupling

Continuing with jacobians

$$[\mathrm{J}KE]_e = (-\Delta x/2, \mathrm{Pe}^{-1}, \mathrm{C}_k^{-1})(\ )\{K\}(-1)[\mathrm{A3110}][\mathrm{RET}, E^{-1}]$$
$$+ (-\Delta x/2)(\ )\{U\}(0)[\mathrm{A3100}][T\mathrm{XY}, E^{-1}]$$
$$+ (\Delta x/2)(\ )\{\ \}(1)[\mathrm{A200}][\ ]$$

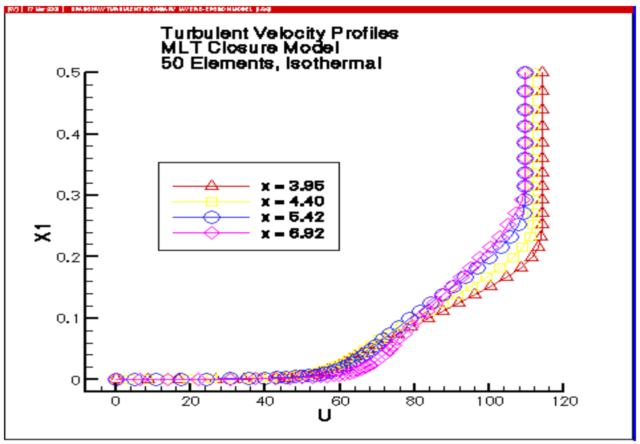
$$[\mathrm{J}EK]_e = (\Delta x, \mathrm{Pe}^{-1}, \mathrm{C}_\varepsilon^{-1})(\ )\{E\}(-1)[\mathrm{A3110}][\mathrm{RET}, K^{-1}]$$
$$+ (-\Delta x/2, \mathrm{C}_\varepsilon^{1})(\mathrm{FE1})\{U\}(0)[\mathrm{A3100}][T\mathrm{XY}, E/K^2]$$
$$+ (\Delta x, \mathrm{C}_\varepsilon^{1})(\mathrm{FE1})\{U\}(0)[\mathrm{A3100}][T\mathrm{XY}, (E/K)^2]$$
$$+ (-\Delta x/2, \mathrm{C}_\varepsilon^{2})(\mathrm{FE2})\{\mathrm{E}\}(1)[\mathrm{A3000}][E/K^2]$$

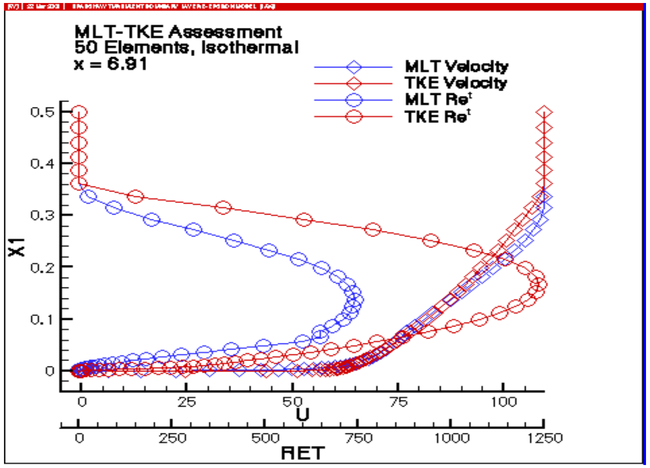
$$[\mathrm{J}EE]_e = (\Delta x/2, \mathrm{Pe}^{-1})(\ )\{\ \}(-1)[\mathrm{A211}][\ ]$$
$$+ (\Delta x/2, \mathrm{Pe}^{-1}, \mathrm{C}_\varepsilon^{1})(\ )\{\mathrm{RET}\}(-1)[\mathrm{A3011}][\ ]$$
$$+ (\Delta x/2, \mathrm{C}_\varepsilon^{2})(\mathrm{FE2})\{E/K\}(1)[\mathrm{A3000}][\ ]$$
$$+ (\Delta x/2, \mathrm{C}_\varepsilon^{2})(\mathrm{FE2})\{E\}(1)[\mathrm{A3000}][K^{-1}]$$

PNS.27 GWS^h + θTS TKE BL, Accuracy, Validation

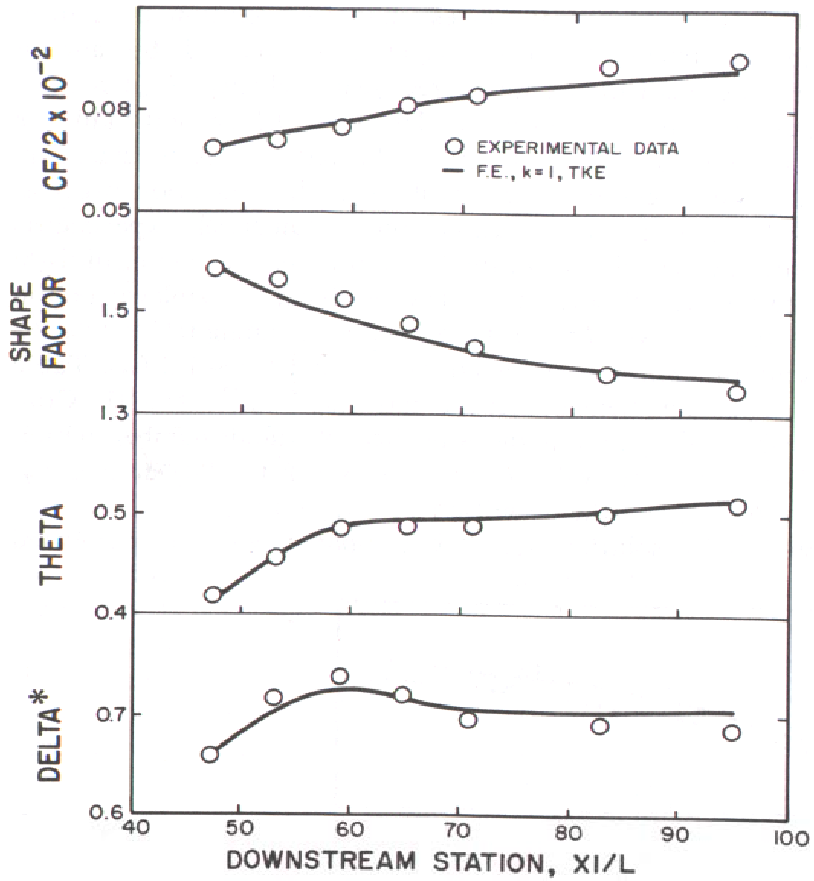
Validation, Bradshaw I 2400 experiment, Re/L≈10^5

MLT, TKE closure solutions





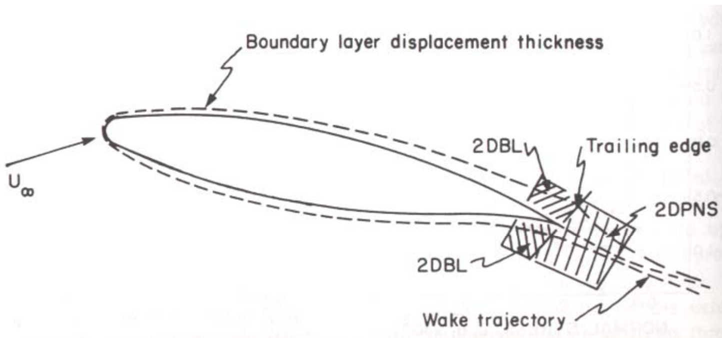
BL integral norm evolutions

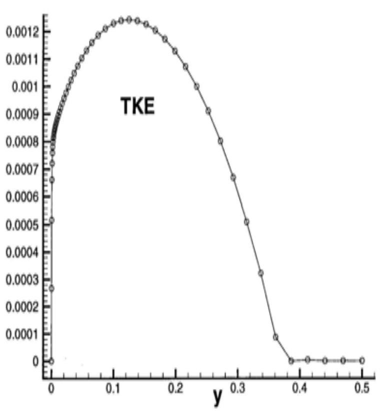


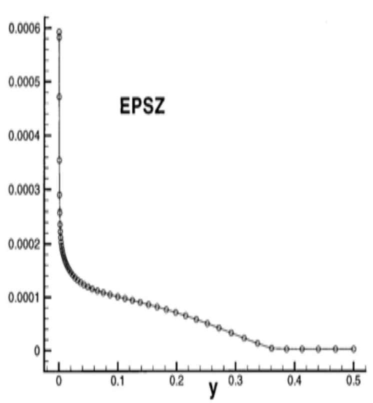
PNS.28 Aerodynamic Trailing Edge Turbulent Wake

Reynolds-ordered PNS PDE+BCs for merging BLs

Problem statement geometry







BL ⇒ PNS theory modifications

D*M* BCs not valid for ODE on {V(*y*)}
  $\Rightarrow \nabla\cdot\mathbf{u} = 0$ is now a differential constraint
$D\mathbf{P}_x$ remains as developed
$D\mathbf{P}_y$ still $O(\delta)$, but must be included for BCs
D*K*, D*E* remain as developed
$\nabla\cdot D\mathbf{P}$ yields pressure Poisson equation
  ⇒ complementary + particular solutions

BL distributions merging at TE

$\max(\partial k / \partial y, \partial \varepsilon / \partial y)\big|_{\partial\Omega} \Rightarrow$ interior to Ω!

requires attention to τ_{ij}

PNS.29 GWS^h + θTS Validation, Turbulent BL ⇒TE Wake

BL ⇒ wake expanded orders for Reynolds stresses

$$\overline{u'}\,\overline{u'} = \overbrace{C_1 k - C_2 C_4 \frac{k^3}{\varepsilon^2}\left(\frac{\partial \overline{u}}{\partial y}\right)^2}^{O(\delta)} - \overbrace{2C_4 \frac{k^2}{\varepsilon}\left(\frac{\partial \overline{u}}{\partial x}\right)}^{O(\delta^2)}$$

$$\overline{v'}\,\overline{v'} = C_3 k - C_2 C_4 \frac{k^3}{\varepsilon^2}\left(\frac{\partial \overline{u}}{\partial y}\right)^2 - 2C_4 \frac{k^2}{\varepsilon}\left(\frac{\partial \overline{u}}{\partial y}\right)$$

$$\overline{w'}\,\overline{w'} = C_3 k$$

$$\overline{u'}\,\overline{v'} = C_2 \frac{k^2}{\varepsilon}\left(\frac{\partial \overline{u}}{\partial y}\right)$$

closure model constants, $C_{01} \approx 2.8$, $C_{01} \approx 0.45$

$$C_1 \equiv \frac{22(C_{01}-1) - 6(4C_{02}-5)}{33(C_{01}-2C_{02})}$$

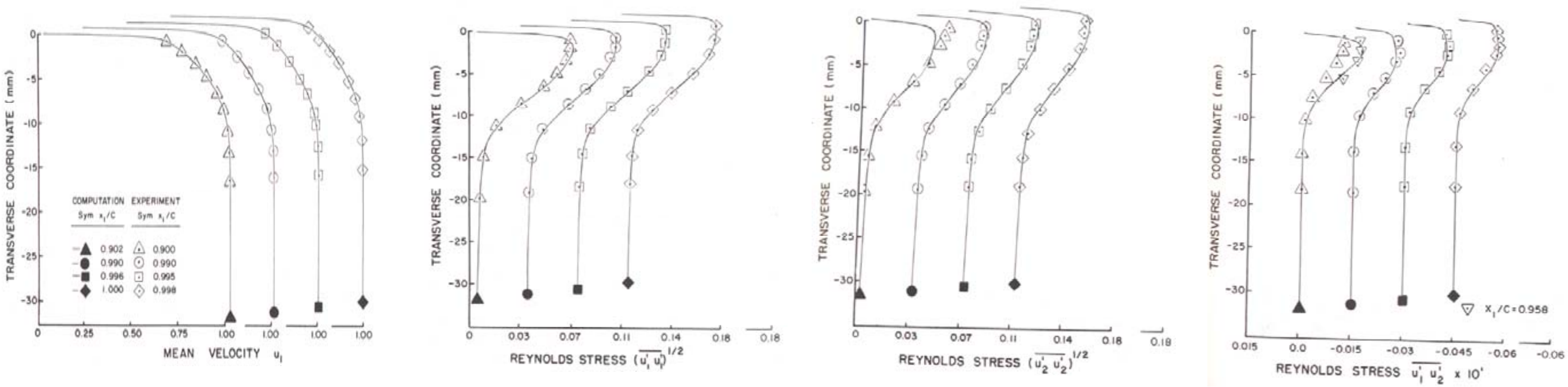
$$C_2 \equiv \frac{4(3C_{02}-1)}{11(C_{01}-2C_{02})}$$

$$C_3 \equiv \frac{22(C_{01}-1) - 12(3C_{02}-1)}{33(C_{01}-2C_{02})}$$

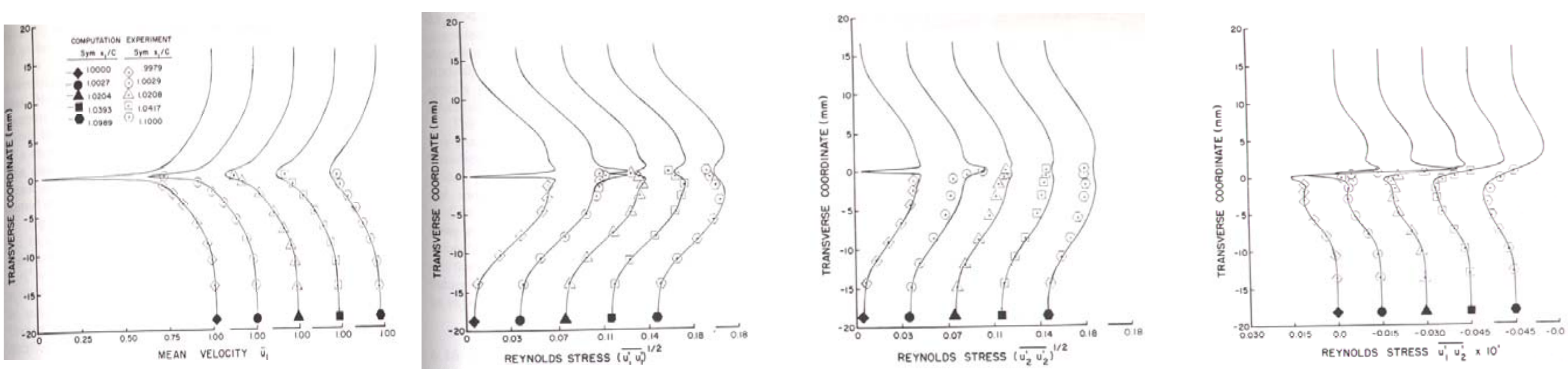
$$C_4 \equiv \frac{44C_{02} - 22C_{01}C_{02} - 128C_{02} - 36C_{02}^2 + 10}{165(C_{01}-2C_{02})^2}$$

PNS.30 GWS^h + θTS Validation, Turbulent BL ⇒TE Wake

GWS^h+θTS BL comparisons, 0.90 ≤ x/chord ≤ 0.998



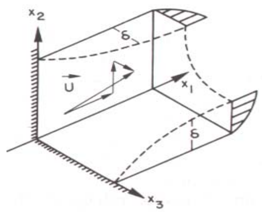
GWS^h+θTS PNS wake comparisons, 1.00 ≤ x/chord ≤ 1.099

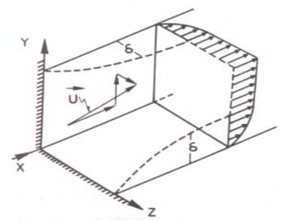


PNS.31 Unidirectional 3-D Aerodynamic Viscous Flows

3-D extensions include juncture region, ducted flows

flow geometries





3D PNS, Favre time-average

$$L(\bar{\rho}) = \frac{\partial}{\partial x_j}\left(\bar{\rho}\tilde{u}_j\right) = 0$$

$$L(\bar{\rho}\tilde{u}_i) = \frac{\partial}{\partial x_j}\left(\bar{\rho}\tilde{u}_i\tilde{u}_j + \bar{p}\delta_{ij} + \bar{\rho}\overline{u_i' u_j'} - \bar{\sigma}_{ij}\right) = 0$$

$$L(\bar{\rho}\tilde{H}) = \frac{\partial}{\partial x_j}\left(\bar{\rho}\tilde{H}\tilde{u}_j + \tilde{u}_i\bar{\sigma}_{ij} + \bar{\rho}\overline{H' u_j'} - \overline{u_i'\sigma_{ij}'} - \bar{g}_j\right) = 0$$

$$\bar{\sigma}_{ij} = \rho\upsilon\frac{\tilde{S}_{ij} - (2/3)\delta_{ij}\tilde{S}_{kk}}{\mathrm{Re}}$$

$$\bar{q}_j = \kappa\frac{\partial\tilde{H}}{\partial x_j}$$

$$\tilde{S}_{ij} \equiv \frac{\partial\tilde{u}_i}{\partial x_j} + \frac{\partial\tilde{u}_j}{\partial x_i}$$

PNS-ordered Reynolds stress tensor

$$\overline{u_1' u_1'} = \overbrace{C_1 k - C_2 C_4\frac{k^3}{\varepsilon^2}\left[\left(\frac{\partial\tilde{u}_1}{\partial x_2}\right)^2 + \left(\frac{\partial\tilde{u}_1}{\partial x_3}\right)^2\right]}^{O(\delta)}\ \overbrace{- 2C_4\frac{k^2}{\varepsilon}\left[\frac{\partial\tilde{u}_1}{\partial x_1}\right]}^{O(\delta^2)}$$

$$\overline{u_2' u_2'} = C_3 k - C_2 C_4\frac{k^3}{\varepsilon^2}\left[\frac{\partial\tilde{u}_1}{\partial x_2}\right]^2 - 2C_4\frac{k^2}{\varepsilon}\left[\frac{\partial\tilde{u}_2}{\partial x_2}\right]$$

$$\overline{u_3' u_3'} = C_3 k - C_2 C_4\frac{k^3}{\varepsilon^2}\left[\frac{\partial\tilde{u}_1}{\partial x_3}\right]^2 - 2C_4\frac{k^2}{\varepsilon}\left[\frac{\partial\tilde{u}_3}{\partial x_3}\right]$$

$$\overline{u_1' u_2'} = -C_4\frac{k^2}{\varepsilon}\left[\frac{\partial\tilde{u}_1}{\partial x_2}\right]^2 - C_2 C_4\frac{k^3}{\varepsilon^2}\left[\frac{\partial\tilde{u}_1}{\partial x_3}\left(\frac{\partial\tilde{u}_2}{\partial x_3}\right)^2 + \left(\frac{\partial\tilde{u}_3}{\partial x_2}\right)^2 + 2\frac{\partial\tilde{u}_1}{\partial x_2}\left(\frac{\partial\tilde{u}_1}{\partial x_1} + \frac{\partial\tilde{u}_2}{\partial x_2}\right)\right]$$

$$\overline{u_1' u_3'} = -C_4\frac{k^2}{\varepsilon}\left[\frac{\partial\tilde{u}_1}{\partial x_3}\right]^2 - C_2 C_4\frac{k^3}{\varepsilon^2}\left[\frac{\partial\tilde{u}_1}{\partial x_2}\left(\frac{\partial\tilde{u}_2}{\partial x_3}\right)^2 + \left(\frac{\partial\tilde{u}_3}{\partial x_2}\right)^2 + 2\frac{\partial\tilde{u}_1}{\partial x_3}\left(\frac{\partial\tilde{u}_1}{\partial x_1} + \frac{\partial\tilde{u}_3}{\partial x_3}\right)\right]$$

$$\overline{u_2' u_3'} = -C_2 C_4\frac{k^3}{\varepsilon^2}\left[\frac{\partial\tilde{u}_1}{\partial x_2}\frac{\partial\tilde{u}_1}{\partial x_3}\right] - C_4\frac{k^2}{\varepsilon}\left(\frac{\partial\tilde{u}_2}{\partial x_3} + \frac{\partial\tilde{u}_3}{\partial x_2}\right)$$

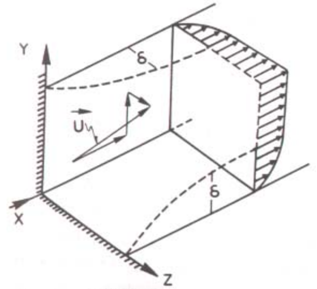
PNS.32 GWS^h + θTS 3D PNS Algorithm, Validation

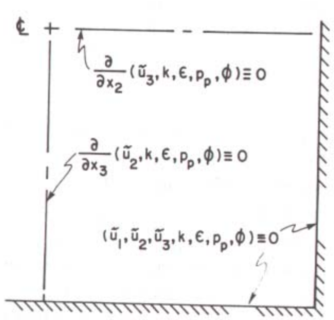
3D PNS algorithm based on a pressure-projection algorithm

$$\mathrm{D}M : \nabla^h \cdot \bar{\rho}\tilde{\mathbf{u}}^h \approx 0 \Rightarrow \mathrm{L}(\phi) = -\nabla^2\phi - \nabla\cdot\bar{\rho}\tilde{\mathbf{u}} = 0 + \mathrm{BCs}$$

$$\nabla\cdot\mathrm{D}\mathbf{P} : \nabla\cdot\mathrm{L}(\tilde{\mathbf{u}}) = 0 \Rightarrow \mathrm{L}(p) = -\nabla\cdot\bar{\rho}\,\nabla p + s(\bar{\rho},\tilde{\mathbf{u}}) = 0 + \mathrm{BCs}$$

3D square duct flow, BCs

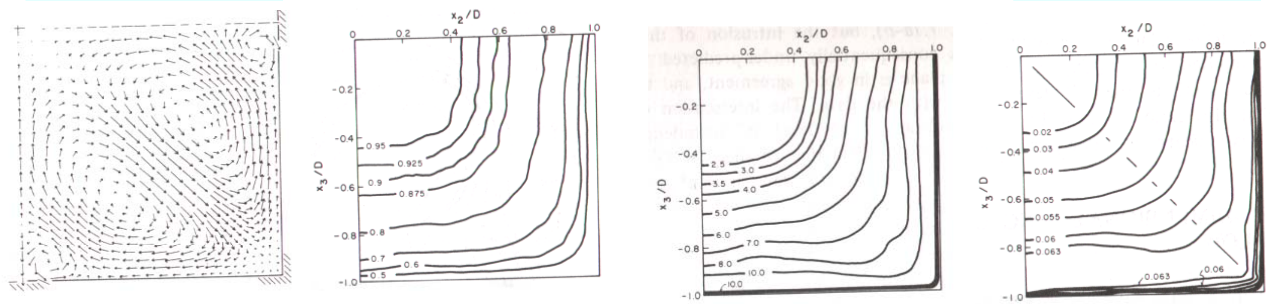




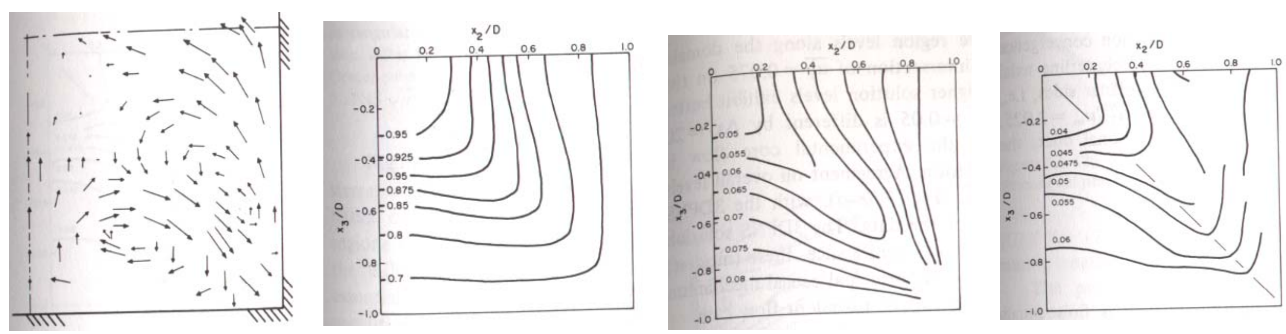
3D PNS GWS^h + θTS solution validation

transverse $\bar{u}_k$ | axial $\bar{u}_1$ | stress $\overline{u_1' u_1'}$ | stress $\overline{u_2' u_2'}$

GWS^h + θTS



experiment



PNS.33 Summary, $GWS^h + \theta TS$ for Parabolic Navier-Stokes

Aerodynamic flows ⇔ weak interaction

streamline shapes
- flowfield is uni-directional
- pressure impressed from farfield

large Reynolds number, $Re/L > 10^6$
- viscous-turbulent effects strictly local
- admits parabolizing steady NS equations

$GWS^h + \theta TS$ algorithm performance for PNS equations

linear asymptotic convergence theory confirmed appropriate, $1 \le k \le 3$ FE bases

$$\left| e^h(n\Delta x) \right|_E \le C h_e^{2k} \left\| \text{data} \right\|_{L2}^2 + C_x \Delta x^3 \left\| q_0 \right\|_{H1}^2$$

GWS^h solution optimality verified in comparison to FVS^h options
MLT & TKE turbulence closure models, including low Re^t
algorithm non-linearities template defined via hypermatrices
validation exercises completed, $n = 2, 3$
- non-linear algebraic Reynolds stress tensor