SVNS.1 Isothermal INS, Streamfuction-Vorticity, *n* = 2

For *n* = 2: $\mathbf{u} = \nabla \times \psi \hat{\mathbf{k}} \; and \; \omega \equiv \nabla \times \mathbf{u} \cdot \hat{\mathbf{k}}$

D*M*: $\nabla \cdot \mathbf{u} = \nabla \cdot \nabla \times \psi \hat{\mathbf{k}} = 0$ identically

$\hat{\mathbf{k}} \cdot \nabla \times \mathrm{DP}$: $\omega_t + \nabla \times \psi \hat{\mathbf{k}} \cdot \nabla \omega - \mathrm{Re}^{-1} \nabla^2 \omega = 0$

kinematics: $\omega = \nabla \times \mathbf{u} \cdot \hat{\mathbf{k}} = \nabla \times \nabla \times \psi \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = -\nabla^2 \psi$

INS D*M* + DP ⇒ **well-posed PDEs + BCs + IC:**

$$\mathrm{L}(\omega) = \partial \omega / \partial t + \nabla \times \psi \hat{\mathbf{k}} \cdot \nabla \omega - \mathrm{Re}^{-1} \nabla^2 \omega = 0$$

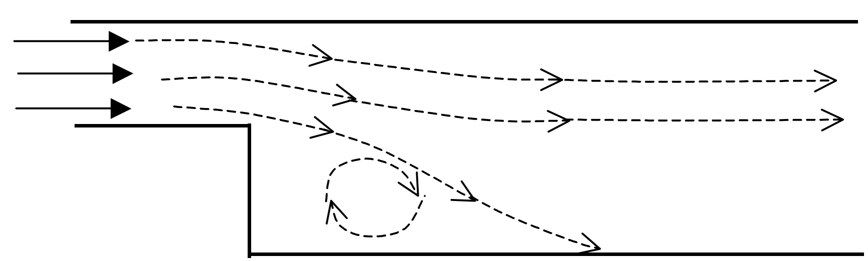
$$\mathrm{L}(\psi) = -\nabla^2 \psi - \omega = 0$$

$\partial \Omega_{\mathrm{in}}$: $\mathbf{u}(y, x_{\mathrm{in}}) \Rightarrow \omega_{\mathrm{in}}, \psi_{\mathrm{in}}$

$\partial \Omega_{\mathrm{out}}$: $\hat{\mathbf{n}} \cdot \nabla(\omega, \psi) = 0$

$\partial \Omega_{\mathrm{wall}}$: $\psi = \psi_w = \text{constant}$

$\hat{\mathbf{n}} \cdot \nabla \omega = f_w(\psi, \omega)$



SVNS.2 GWS^h, Streamfunction-Vorticity INS, $n = 2$

Galerkin weak statements:

for $q^N(x,y,t) \equiv \sum_\alpha \Psi_\alpha(\mathbf{x}) Q_\alpha(t), \quad q = \{\omega, \psi\}^T$

$$\mathrm{GWS}^N(\omega) \equiv \int_\Omega \Psi_\beta(\mathbf{x}) \mathrm{L}(\omega^N) d\tau = \int_\Omega \Psi_\beta \left[\partial\omega^N / \partial t + \nabla \times \psi^N \hat{\mathbf{k}} \cdot \nabla\omega^N - \mathrm{Re}^{-1} \nabla^2 \omega^N \right] d\tau$$

$$= [\mathrm{MASS}] d\{\mathrm{OMG}\}/dt + \{\mathrm{RES}(\psi^N, \mathrm{Re})\} + \mathrm{BCs} = \{0\}$$

$$\mathrm{GWS}^N(\psi) = \int_\Omega \Psi_\beta L(\psi^N) d\tau = \int_\Omega \Psi_\beta (-\nabla^2 \psi^N - \omega^N) d\tau = [\mathrm{DIFF}]\{\mathrm{PSI}\} - [\mathrm{MASS}]\{\mathrm{OMG}\} = \{0\}$$

GWS^N + θTS produces algebraic statements

$$\{\mathrm{FOMG}\} = [\mathrm{MASS}]\{\Delta\mathrm{OMG}\} + \Delta t\left([\mathrm{CONV}(\psi^N)] + \mathrm{Re}^{-1}[\mathrm{DIFF}]\right)\{\mathrm{OMG}\}_\theta + \mathrm{BCs}$$

$$\{\mathrm{FPSI}\} = [\mathrm{DIFF}]\{\mathrm{PSI}\} - [\mathrm{MASS}]\{\mathrm{OMG}\} + \mathrm{BCs}$$

thus: $\mathrm{GWS}^N \Rightarrow \mathrm{GWS}^h = \mathrm{S}\{\mathrm{WS}\}_e \equiv \{0\}$

$$\{\mathrm{WS}(\omega^h)\}_e = [\mathrm{B200}]_e\{\Delta\mathrm{OMG}\} + \Delta t(\mathrm{Re}^{-1}[\mathrm{B2}KK]_e\{\mathrm{OMG}\}_e + \{\mathrm{PSI}\}_e^T[\mathrm{B3}K0K]_e\{\mathrm{OMG}\}_e$$

$$+ \mathrm{Re}^{-1}[\mathrm{A200}]_e\{f_w(\psi^h, \omega^h\}_e)_\theta$$

$$\{\mathrm{WS}(\psi^h)\}_e = [\mathrm{B2}KK]\{\mathrm{PSI}\}_e - [\mathrm{B200}]_e\{\mathrm{OMG}\}_e + [\mathrm{A200}]_e\{\mathrm{U}_w\}_e$$

SVNS.3 GWS^h Details, Streamfunction-Vorticity INS

GWS^h for ω^h involves $\nabla \times \psi \hat{\mathbf{k}} \cdot \nabla \omega$

$$\nabla \times \psi \hat{\mathbf{k}} \cdot \nabla \omega = \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} \Rightarrow [\mathrm{B3}K0K]_e = \int_{\Omega_e} \left[\frac{\partial \{N\}}{\partial y}\{N\}\frac{\partial \{N\}^T}{\partial x} - \frac{\partial \{N\}}{\partial x}\{N\}\frac{\partial \{N\}^T}{\partial x} \right] \mathrm{d}x\mathrm{d}y$$

$$= [\mathrm{B3Y0X}]_e - [\mathrm{B3X0Y}]_e$$

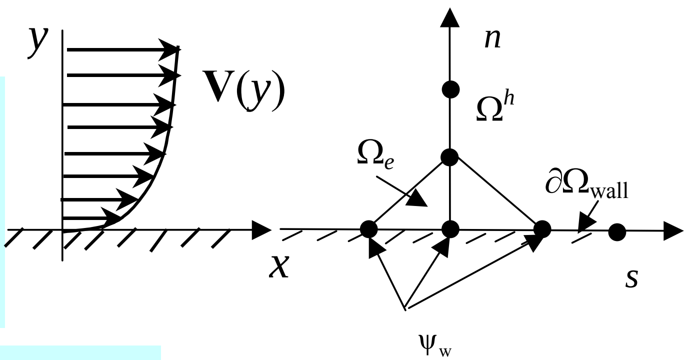
Vorticity Robin BC generated from kinematics and TS

$$\mathrm{L}(\psi) = -\nabla^2\psi - \omega = \frac{-\partial^2\psi}{\partial s^2} - \frac{\partial^2\psi}{\partial n^2} - \omega = 0 \Rightarrow \frac{\mathrm{d}^2\psi}{\mathrm{d}n^2} = -\omega \Big|_{\partial\Omega_{\mathrm{wall}}}$$

TS:

$$\psi(\Delta n) = \psi_{\mathrm{w}} + \frac{\mathrm{d}\psi}{\mathrm{d}n}\Big|_{\mathrm{w}} \Delta n + \frac{\mathrm{d}^2\psi}{\mathrm{d}n^2}\Big|_{\mathrm{w}} \frac{\Delta n^2}{2} + \frac{\mathrm{d}^3\psi}{\mathrm{d}n^3}\Big|_{\mathrm{w}} \frac{\Delta n^3}{6} + O(\Delta n^4)$$

$$= \psi_{\mathrm{w}} + \mathrm{U}_{\mathrm{w}}\Delta n - \omega_{\mathrm{w}}\Delta n^2/2 - (\mathrm{d}\omega/\mathrm{d}n)_{\mathrm{w}}\Delta n^3/6$$



BC:

$$\ell(\omega) = \nabla\omega \cdot \hat{\mathbf{n}} + (3/\Delta n)\omega - (6/\Delta n^2)\mathrm{U}_{\mathrm{w}} + (6/\Delta n^3)\Delta\psi_{\mathrm{w}} = 0$$

SVNS.4 Newton Template, INS (ω^h, ψ^h) GWS^h

GWS^h + θTS $\Rightarrow$ Newton iteration algorithm

$$[\mathrm{JAC}]\{\delta Q\}^{p+1} = -\{\mathrm{FQ}\}^p \Leftrightarrow \mathrm{S}_e\left([\mathrm{JAC}]_e\right)\{\delta Q\}^{p+1} = -\mathrm{S}_e\left(\{\mathrm{FQ}\}_e\right)$$

Template pseudo code notation convention

$$\{\mathrm{WS}(\,\cdot\,)\}_e \equiv (\mathrm{const})\ (\mathrm{avg})_e\{\mathrm{dist}\}_e(\mathrm{metric;det})_e[\mathrm{Matrix}]\{Q \text{ or data}\}_e$$

Diffusion term in $\{\mathbf{WS}\}_e$, $n = 2, 3$, $1 \le (I, J, K) \le n, n + 1$

$$[\mathrm{DIFF}]_e = \mathrm{CONST\ COND}_e\ \mathrm{ET}JI_e\ \mathrm{ET}KI_e\ \mathrm{DET}_e^{-1}\ [\mathrm{M2}JK]$$

single array metric data ordering, $\{N^+(\eta)\}$, $\{N(\zeta)\}$

$$n = 2, 3: \quad \mathrm{ET}KI \Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{\mathrm{TP}}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{\mathrm{NC}}; \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{\mathrm{TP}}, \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}_{\mathrm{NC}}$$

SVNS.5 INS (ω^h, ψ^h) GWS^h $\{FQ\}_e$ Template, $\{N_1^+(\eta)\}$

$$
\begin{aligned}
\{\mathrm{FOMG}\}_e &= [\mathrm{B200}]_e\{\Delta\mathrm{OMG}\}_e + \Delta t\Big((\{\mathrm{PSI}\}_e^T[\mathrm{B3}K0K]_e + \mathrm{Re}^{-1}[\mathrm{B2}KK]_e)\{\mathrm{OMG}\}_e\Big)_\theta \\
&= \ell_e[\mathrm{B200}]\{\Delta\mathrm{OMG}\}_e + \Delta t\big[\{\mathrm{PSI}\}_e^T(12\cdot 21 - 11\cdot 22)_e \mathrm{DET}_e^{-1}[\mathrm{B3102}]\{\mathrm{OMG}\}_e \\
&\qquad + \{\mathrm{PSI}\}_e^T(22\cdot 11 - 21\cdot 12)_e \mathrm{DET}_e^{-1}[\mathrm{B3201}]\{\mathrm{OMG}\}_e \\
&\qquad + \mathrm{Re}^{-1}(\mathrm{E}JI\cdot\mathrm{E}KI)_e[\mathrm{B2}JK]\{\mathrm{OMG}\} + \{\mathrm{BC}\}_e\big]_\theta \\
&= (\ \ )(\ \ )\{\ \ \}(0;1)[\mathrm{B200}]\{\mathrm{OMGP}-\mathrm{OMGN}\} \\
&\quad + \big[(\Delta t)(\ \ )\{\mathrm{PSI}\}(23-14;-1)[\mathrm{B3102}]\{\mathrm{OMG}\} \\
&\quad + (\Delta t)(\ \ )\{\mathrm{PSI}\}(41-32;-1)[\mathrm{B3201}]\{\mathrm{OMG}\} \\
&\quad + (\Delta t,\mathrm{Re}^{-1})(\ \ )\{\ \ \}\big[(1122;-1)[\mathrm{B211}] + (3344;-1)[\mathrm{B222}] \\
&\qquad + (1324;-1)[\mathrm{B221}] + (3142;-1)[\mathrm{B212}]\big]\{\mathrm{OMG}\} \\
&\quad + (\Delta t,\mathrm{Re}^{-1})(\ \ )\{\ \ \}(0;1)[\mathrm{A200}]\{\ell(\omega)\}\big]_\theta
\end{aligned}
$$

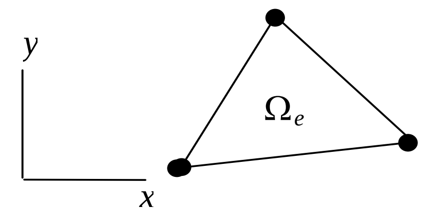
$$
\begin{aligned}
\{\mathrm{FPSI}\}_e &= [\mathrm{B2}KK]_e\{\mathrm{PSI}\}_e - [\mathrm{B200}]_e\{\mathrm{OMG}\}_e + \{\mathrm{BC}\}_e \\
&= (\ \ )(\ \ )\{\ \ \}\big[(1122;-1)[\mathrm{B211}] + (3344;-1)[\mathrm{B222}] \\
&\qquad + (1324;-1)[\mathrm{B221}] + (3142;-1)[\mathrm{B212}]\big]\{\mathrm{PSI}\} \\
&\quad + (-)(\ \ )\{\ \ \}(0;1)[\mathrm{B200}]\{\mathrm{OMG}\} + (\ \ )(\ \ )\{\ \ \}(0;1)[\mathrm{A200}]\{\mathrm{U_w}\}
\end{aligned}
$$

SVNS.5A INS (ω^h, ψ^h) GWS^h $\{N_1\}$ Bases Convection Matrices

GWS^h for $\nabla \times \psi \hat{\mathrm{k}} \cdot \nabla \omega$ generates skew-symmetric hypermatrix

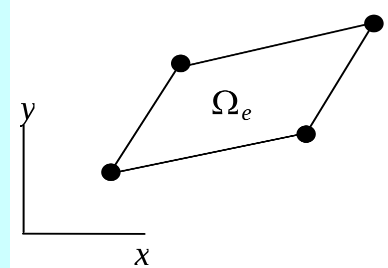
for $\{N_1(\zeta)\}$, any Ω_e:

$$[\mathrm{B3}K0K\mathrm{L}] = \frac{1}{6}\begin{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} & \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} & \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix} \\ \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} & \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} & \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix} \\ \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} & \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} & \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix} \end{bmatrix}$$



for $\{N_1^{+}(\zeta)\}$, parallelogram Ω_e:

$$[\mathrm{B3}K0K\mathrm{BL}] = \frac{1}{\mathrm{X}}\begin{bmatrix} \begin{Bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{Bmatrix} & \begin{Bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{Bmatrix} & \begin{Bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{Bmatrix} & \begin{Bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{Bmatrix} \\ \begin{Bmatrix} 0 \\ 2 \\ -1 \\ -1 \end{Bmatrix} & \begin{Bmatrix} -2 \\ 0 \\ 2 \\ 0 \end{Bmatrix} & \begin{Bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{Bmatrix} & \begin{Bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{Bmatrix} \\ \begin{Bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{Bmatrix} & \begin{Bmatrix} -1 \\ 0 \\ 2 \\ -1 \end{Bmatrix} & \begin{Bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{Bmatrix} & \begin{Bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{Bmatrix} \\ \begin{Bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{Bmatrix} & \begin{Bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} & \begin{Bmatrix} -1 \\ -1 \\ 0 \\ 2 \end{Bmatrix} & \begin{Bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{Bmatrix} \end{bmatrix}$$



note: for both basis forms

$$\{\mathrm{WS}(\nabla \times \Psi^h \cdot \nabla)\}_e \neq f(\det_e)$$

SVNS.6 INS (ω^h, ψ^h) GWS^h Template Completion

Newton jacobian formed via differentiation

$$[\mathrm{JAC}]_e \equiv \frac{\partial\{\mathrm{F}Q\}_e}{\partial\{Q\}_e} = \begin{bmatrix} [\mathrm{J}\Omega\Omega] & , & [\mathrm{J}\Omega\psi] \\ [\mathrm{J}\psi\Omega] & , & [\mathrm{J}\psi\psi] \end{bmatrix}_e$$

Jacobian template pseudo-code, compressed essence

$$[\mathrm{J}\Omega\Omega]_e \equiv \frac{\partial\{\mathrm{FOMG}\}_e}{\partial\{\mathrm{OMG}\}_e} = [\mathrm{B200}]_e + (\theta\Delta t, \mathrm{Re}^{-1})(\ )\{\ \}(-1)[\mathrm{B2}KK]_e[\ ]$$

$$+ (\theta\Delta t)(\ )\{\mathrm{PSI}\}_e(-1)[\mathrm{B3}K0K]_e[\ ] + (\theta\Delta t, 3/\Delta n, \mathrm{Re}^{-1})(\ )\{\ \}(1)[\mathrm{A200}][\ ]$$

$$[\mathrm{J}\Omega\psi]_e \equiv \frac{\partial\{\mathrm{FOMG}\}_e}{\partial\{\mathrm{PSI}\}_e} = (\theta\Delta t)(\ )\{\mathrm{OMG}\}_e(-1)[\mathrm{B3}K0K\mathrm{T}]_e[\ ] - (\theta\Delta t, 6/\Delta n^3, \mathrm{Re}^{-1})(\ )\{\ \}(1)[\mathrm{A200}]$$

$$[\mathrm{J}\psi\Omega]_e \equiv \frac{\partial\{\mathrm{FPSI}\}_e}{\partial\{\mathrm{OMG}\}_e} = (-1)(\ )\{\ \}(1)[\mathrm{B200}][\ ]$$

$$[\mathrm{J}\psi\psi]_e \equiv \frac{\partial\{\mathrm{FPSI}\}_e}{\partial\{\mathrm{PSI}\}_e} = (\ )(\ )\{\ \}(-1)[\mathrm{B2}KK]_e[\ ]$$

SVNS.7 INS Intrinsic Variable Recovery from (ω^h, ψ^h)

Velocity vector field computable via 2 kinematic relationships

vorticity: $\nabla \times \omega \hat{\mathbf{k}} = \nabla \times \nabla \times \mathbf{u} \cdot \hat{\mathbf{k}} = \nabla^2 \mathbf{u}$

$$\text{PDE}: \quad \mathrm{L}(\mathbf{u}) = -\nabla^2 \mathbf{u} + \nabla \times \omega \hat{\mathbf{k}} = \mathbf{0}$$

$$\text{BCs}: \quad \ell(\mathbf{u}) = \mathrm{a}\mathbf{u} + \mathrm{b}\nabla \mathbf{u} \cdot \hat{\mathbf{n}} = \mathbf{0}$$

$$\text{GWS}^h: \ \{\text{F}\mathbf{U}\} = [\text{B2}KK]\{\text{U}J\} \pm [\text{B20}I]\{\text{OMG}\}, \ J \neq I$$

streamfunction: $\mathrm{L}(\mathbf{u}) = \mathbf{u} - \nabla \times \psi \hat{\mathbf{k}} = \mathbf{0}$

$$\text{GWS}^h: \ \{\text{F}\mathbf{U}\} = [\text{B200}]\{\text{U}J\} \pm [\text{B20}K]\{\text{PSI}\}, \ J \neq K$$

$$\text{data}: \ \mathbf{u} = \mathbf{0} \ \text{ on no-slip walls}$$

GWS^h template pseudo-code essence

$$\{\text{FUV1}\}_e = (\ )(\ )\{\ \}(\ ;-1)[\text{B2}KK]_e\{\text{UV1}\}_e + (\ )(\ )\{\ \}(\ ;0)[\text{B202}]_e\{\text{OMG}\}_e$$

$$\{\text{FUV2}\}_e = (\ )(\ )\{\ \}(\ ;-1)[\text{B2}KK]_e\{\text{UV2}\}_e + (-)(\ )\{\ \}(\ ;0)[\text{B201}]_e\{\text{OMG}\}_e$$

$$\{\text{FUS1}\}_e = (\ )(\ )\{\ \}(0\ ;1)[\text{B200}]\{\text{US1}\}_e + (\ )(\ )\{\ \}(\ ;0)[\text{B202}]_e\{\text{PSI}\}_e$$

$$\{\text{FUS2}\}_e = (\ )(\ )\{\ \}(0\ ;1)[\text{B200}]\{\text{US2}\}_e + (-)(\ )\{\ \}(\ ;0)[\text{B201}]_e\{\text{PSI}\}_e$$

SVNS.8 Pressure Recovery from (ω^h, ψ^h)

A well-posed laplacian PDE is generated via ∇·DP

$$\mathbf{DP}:\ \mathrm{L}(u_i) = \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}\left(u_j u_i + \frac{1}{\rho_0} p\delta_{ij} - \frac{1}{\mathrm{Re}}\frac{\partial u_i}{\partial x_j}\right) + \frac{\mathrm{Gr}}{\mathrm{Re}^2}\Theta \hat{g}_i = 0$$

$$\nabla\cdot\mathbf{DP}:\ \frac{\partial}{\partial x_i}\mathrm{L}(u_i) = \frac{-\partial}{\partial x_i}\left(\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + u_j\frac{\partial u_i}{\partial x_j} + \frac{\mathrm{Gr}}{\mathrm{Re}^2}\hat{g}_i\Theta\right) = 0$$

$$\mathrm{GWS}^N(\nabla\cdot\mathbf{DP}) = \int_\Omega \Psi_\beta \frac{\partial}{\partial x_i}\mathrm{L}(u_i)\mathrm{d}\tau \equiv 0$$

$$= \int_\Omega \frac{\partial \Psi_\beta}{\partial x_i}\left(\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + u_j\frac{\partial u_i}{\partial x_j} + \frac{\mathrm{Gr}}{\mathrm{Re}^2}\hat{g}_i\Theta\right)\mathrm{d}\tau - \oint_{\partial\Omega}\Psi\left(\frac{1}{\mathrm{Re}}\frac{\partial^2 u_i}{\partial x_j^2} - \frac{\partial u_i}{\partial t}\right)\cdot\hat{n}_i \mathrm{d}\sigma$$

$$\mathrm{GWS}^h \equiv \{\mathrm{F}P\} = \frac{1}{\rho_0}[\mathrm{B2}KK]\{P\} + \{\mathrm{U}J\}^T[\mathrm{B30}IJ]\{\mathrm{U}I\} + \frac{\mathrm{Gr}}{\mathrm{Re}}[\mathrm{B20}I]\{T\}\hat{g}_i$$

$$+\frac{1}{\Delta t}[\mathrm{A200}]\{\Delta\mathrm{U}I\}\hat{n}_i + \frac{1}{\mathrm{Re}}[\mathrm{A211}]\{\mathrm{U}I\}\cdot\hat{n}_i - \mathrm{BCs}$$

$$\mathbf{note}:\ \{\mathrm{WS(BC)}\}_e = -\mathrm{Re}^{-1}\int_{\partial\Omega_e}\{N\}\nabla^2\mathbf{u}_e\cdot\hat{\mathbf{n}}_e\mathrm{d}\sigma = \mathrm{Re}^{-1}\int_{\partial\Omega_e}\nabla\{N\}\cdot\nabla\{N\}^T\mathrm{d}\sigma\{\mathrm{U}IN I\}_e - \{N\}\nabla\mathbf{u}_e\cdot\hat{\mathbf{n}}\big|_{\mathrm{wall}}$$

SVNS.9 GWS^h Algorithm Benchmark for (ω^h, ψ^h)

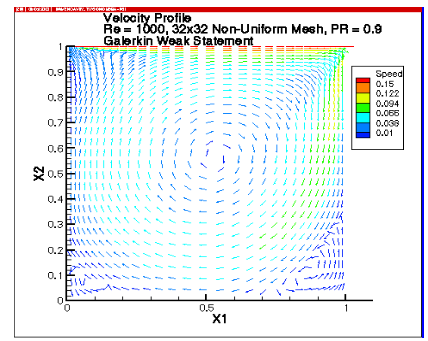
Classical 2-D benchmark, steady-state driven cavity

state variable: $q(\mathbf{x}, t) = \{\omega^h, \psi^h, \mathbf{us}^h, \mathbf{up}^h, p^h\}^T$

BCs:

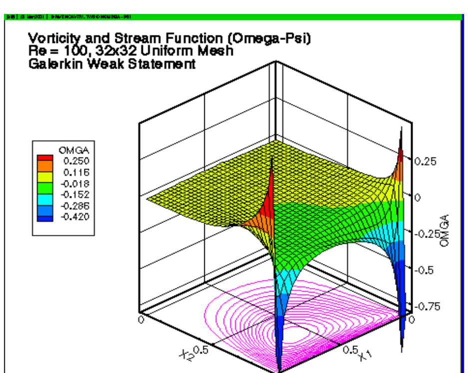
- all walls no slip
- $\psi_w = 0$ all around
- $u(x, y = b) = U_{lid}, v = 0$
- no inflow or outflow
- no pressure BCs

Problem geometry

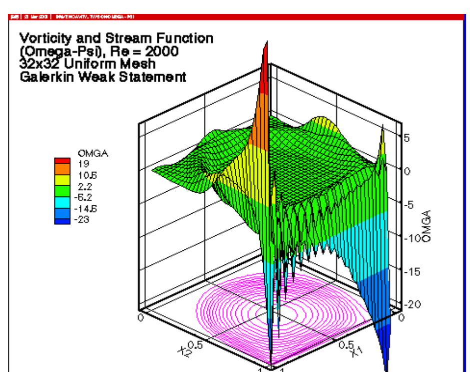


Computer lab 4, accuracy/stability as f (Re, Ω^h)

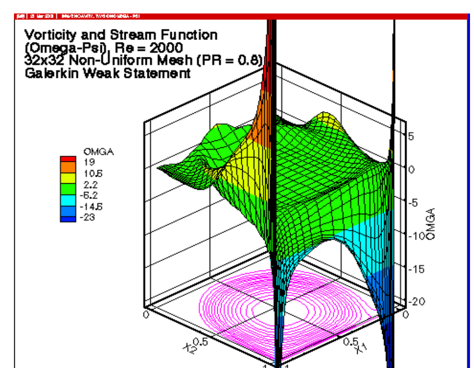
Re = 100, Ω^h uniform



Re = 2000, Ω^h uniform



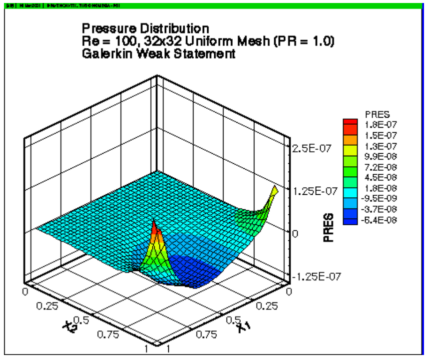
Re = 2000, Ω^h non-uniform



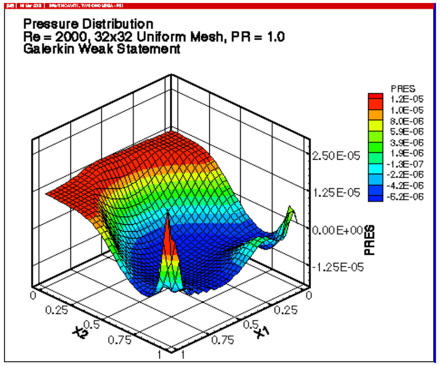
SVNS.10 Driven Cavity, Auxiliary GWS^h Performance

Pressure distributions, PCG solver, no Dirichet BCs

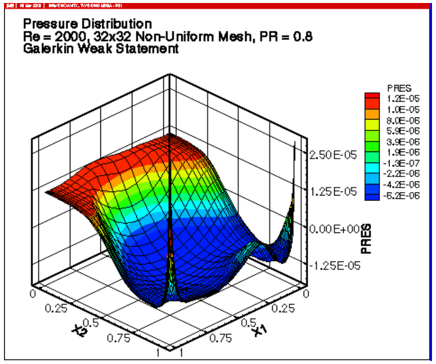
Re = 100, Ω^h uniform



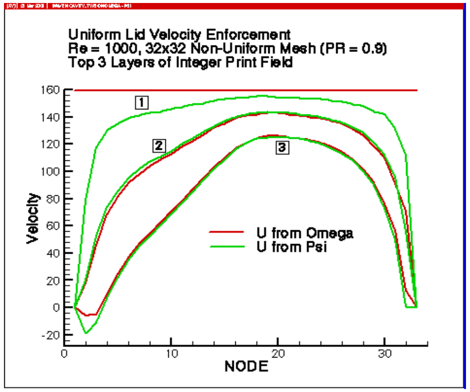
Re = 2000, Ω^h uniform



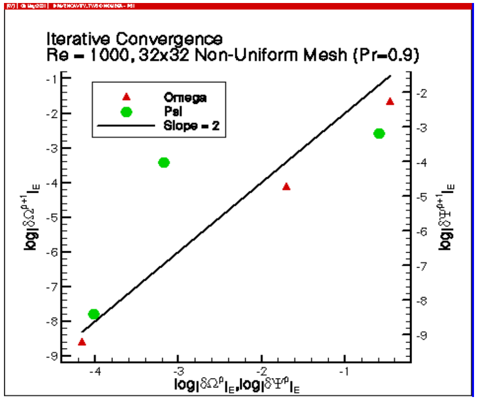
Re = 2000, Ω^h non-uniform



u velocity distributions



Newton iteration convergence

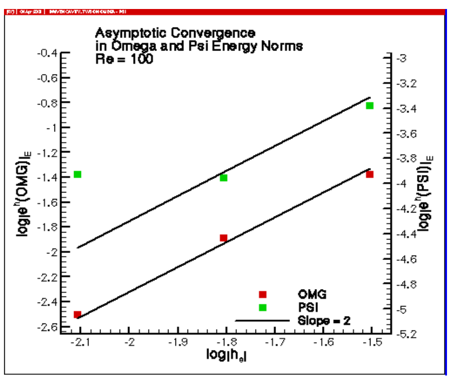


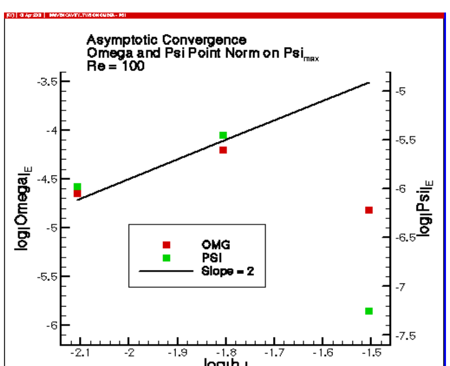
SVNS.11 GWS^h INS (ω^h, ψ^h), Accuracy, Convergence, Optimality

Theoretical asymptotic error estimate

$$\left| e^h(n\Delta t) \right|_E \le C h^{2\gamma} \left\| \text{data} \right\|^2_{L2} + C_t \Delta t^{f(\theta)} \left\| Q_0 \right\|^2_{H1}, \; \gamma = \min(k, r-1)$$

Driven cavity, Re = 100, $16^2 \le M \le 64^2$ uniform Ω^h, steady-state





GWS^h optimality, $100 \le \text{Re} \le 1000$

Extremum Nodal Psi and Omega

Re	Ghia, Ghia and Shin (1982)		TWS^h (Noronha, 1989)		Re	GWS^h		TWS^h (Kolesnikov, 2000)	
	$\lvert Psi\rvert_{max}$	|Omega|	$\lvert Psi\rvert_{max}$	|Omega|		$\lvert Psi\rvert_{max}$	|Omega|	$\lvert Psi\rvert_{max}$	|Omega|
100	0.103423	3.16646	0.10378	3.2990	100	0.100544	3.33963	----	----
400	0.113909	2.29469	0.11473	2.3212		----	----	----	----
1000	0.117929	2.04908	0.11905	2.1107	1000	0.117060	1.98391	----	----
3200	0.120377	1.98860	----	----	2000	0.119834	1.89343	0.127008	2.08944

SVNS.12 TWS^h for INS (ω^h, ψ^h), Stability, Monotonicity

Driven cavity GWS^h solutions suffer dispersion error at larger Re

$$\mathbf{DP}: \ \mathrm{L}(\omega) = \partial\omega/\partial t + \mathbf{u}\cdot\nabla\omega - \mathrm{Re}^{-1}\nabla\omega = 0$$

$$\cong \omega_t + \mathbf{f}_{\mathbf{x}} + O(\varepsilon), \quad \mathbf{f} = \mathbf{u}\omega$$

$$\mathrm{TS}: \quad \omega_{n+1} = \omega_n + \Delta t \left.\frac{\partial\omega}{\partial t}\right|_n + \frac{\Delta t^2}{2}\frac{\partial^2\omega}{\partial t^2} + O(\Delta t^3)$$

$$\omega_t = -\mathbf{f}_{\mathbf{x}} = -\partial f_j/\partial x_j$$

$$\omega_{tt} = -(\partial f_j/\partial x_j)_t = -\partial(\partial f_j/\partial t)/\partial x_j$$

$$= \frac{\partial}{\partial x_j}\left(\alpha\frac{\partial f_j}{\partial\omega}\frac{\partial\omega}{\partial t} + \beta\frac{\partial f_j}{\partial\omega}\frac{\partial f_k}{\partial\omega}\frac{\partial\omega}{\partial x_k}\right)$$

substituting into TS, taking lim ($\Delta t \Rightarrow \varepsilon > 0$) yields

$$\mathbf{DP}^m: \quad \mathrm{L}^m(\omega) = \mathrm{L}(\omega) - \frac{\Delta t}{2}\frac{\partial}{\partial x_j}\left(\alpha u_j\frac{\partial\omega}{\partial t} + \beta u_j u_k\frac{\partial\omega}{\partial x_k}\right) + O(\Delta t^2)$$

where: α term affects time evolution

β term imparts a tensor *numerical* diffusion mechanism

SVNS.13 TWS^h + θTS (ω^h, ψ^h) INS Algorithm

Limiting attention to the steady-state solution

$$\mathrm{GWS}^N \Rightarrow \mathrm{TWS}^N \equiv \int_\Omega \Psi_\beta(x)\mathrm{L}^m(\omega^N)d\tau \equiv 0\,,\ \forall\beta$$

$$= \int_\Omega \Psi_\beta(\mathbf{x})\left[\mathrm{L}(\omega^N) - \frac{\beta\Delta t}{2}\frac{\partial}{\partial x_j}\left(u_j u_k \frac{\partial \omega^N}{\partial x_k}\right)\right]d\tau$$

$$= [\mathrm{MASS}]\{\mathrm{OMG}\}' + \{\mathrm{RES}(\cdot,\beta)\}$$

$$\mathrm{TWS}^N + \theta\mathrm{TS} \equiv \{\mathrm{FOMG}\} = [\mathrm{MASS}]\{\Delta\mathrm{OMG}\} + \Delta t\{\mathrm{RES}(\cdot,\beta)\}\big|_\theta = \{0\}$$

Template pseudo-code essence

$$\mathrm{TWS}^N \Rightarrow \mathrm{TWS}^h + \theta\mathrm{TS} = \mathrm{S}_e\{\mathrm{WS}\}_e = \{0\}$$

$$\{\mathrm{FOMG}\}_e = [\mathrm{B200}]_e\{\Delta\mathrm{OMG}\}_e + \Delta t\{\mathrm{PSI}\}_e^T[\mathrm{B3}K0K]_e\{\mathrm{OMG}\}_e\big|_\theta$$
$$+\Delta t\left(\mathrm{Re}^{-1}[\mathrm{B2}KK]_e + (\beta\Delta t/2)\{\mathrm{UJUK}\}_e^T[\mathrm{B30}JK]_e\right)\{\mathrm{OMG}\}_e\big|_\theta + \{\mathrm{BC}\}_e$$

$$\{\mathrm{FPSI}\}_e = [\mathrm{B2}KK]_e\{\mathrm{PSI}\}_e - [\mathrm{B200}]_e\{\mathrm{OMG}\}_e + \{\mathrm{BC}\}_e$$

SVNS.14 TWS^h (β) (ω^h, ψ^h) INS Template Options

TS generated the β term leading to

$$\{\mathrm{WS}(\beta)\}_e = (\beta\Delta t/2)\int_{\Omega_e} \frac{\partial\{N\}}{\partial x_j} u_j^e u_k^e \frac{\partial\{N\}^T}{\partial x_k} \mathrm{d}\tau\{\mathrm{OMG}\}_e$$

$$= (\beta, \Delta t, 1/2)(\quad)\{\mathrm{U}J, \mathrm{U}K\}\{\mathrm{E}JL, \mathrm{E}KI; -1)[\mathrm{B30}LI]\{\mathrm{OMG}\}$$

one can define a local time scale

$$\Delta t \approx h/|\mathbf{U}|, \quad \{\mathrm{U}J/|\mathbf{U}|\} \equiv \{\mathrm{UJU}\}\ \text{(unit vector)}$$

$$\{\mathrm{WS}(\beta)\}_e = (\beta/2)(\quad)\{\mathrm{UJU}, \mathrm{U}K\}(\mathrm{E}JL, \mathrm{E}KI; -0.5)[\mathrm{B30}LI]\{\mathrm{OMG}\}$$

TS operation on x rather than *t* for steady-state INS leads to

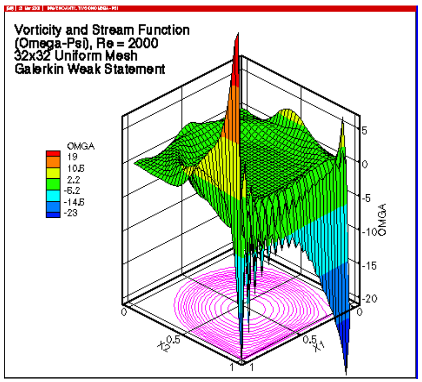
$$\{\mathrm{WS}(\beta)\}_e = (\beta, h^2\,\mathrm{Re}/12)\int_{\Omega_e} \frac{\partial\{N\}}{\partial x_j} u_j^e u_k^e \frac{\partial\{N\}^T}{\partial x_k} \mathrm{d}\tau\{\mathrm{OMG}\}_e$$

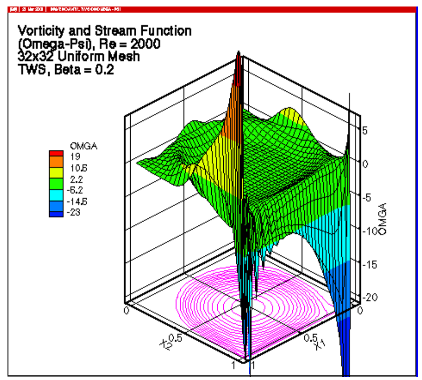
$$= (\beta, \mathrm{Re}/3)(\quad)\{\mathrm{U}J, \mathrm{U}K\}(\mathrm{E}JL, \mathrm{E}KI; 0)[\mathrm{B30}LI]\{\mathrm{OMG}\}$$

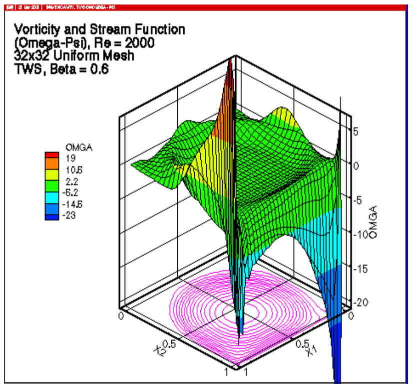
$$\text{where}: \beta = (0, 1), \quad h^2 \approx 4\det_e$$

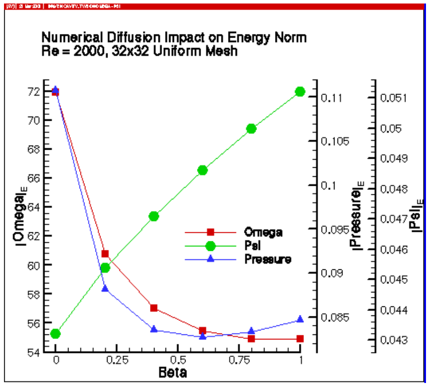
SVNS.15 TWS^h + θTS (ω^h, ψ^h) INS Algorithm Stability

TWS^h (β) solutions, uniform M = 32^2 Ω^h, Re = 2000, $0 \leq \beta \leq 0.8$

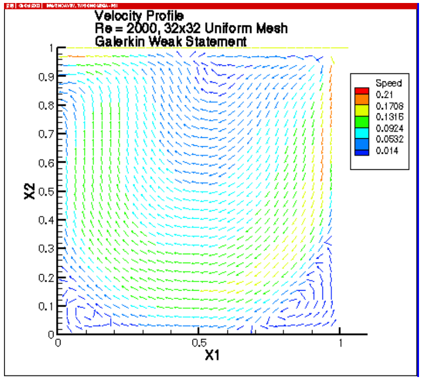


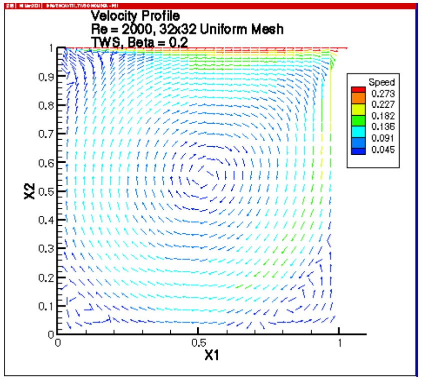


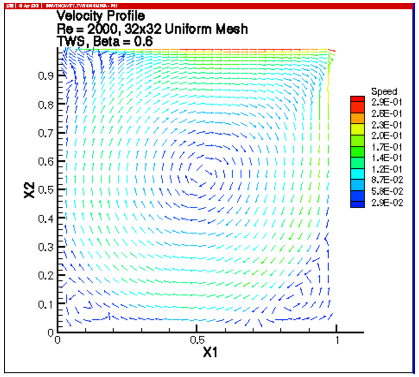


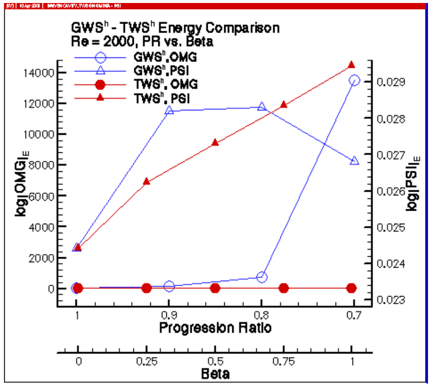


non-uniform Ω^h Pr vs. β



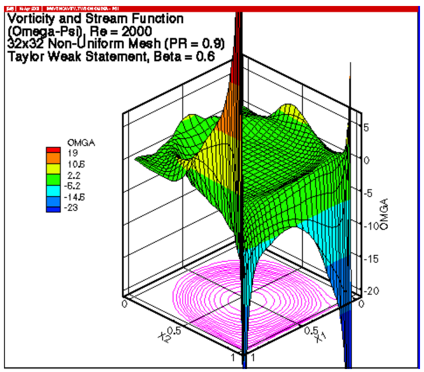


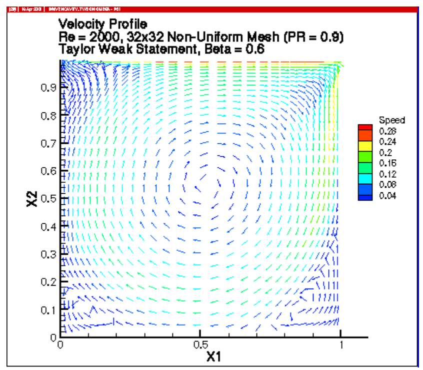


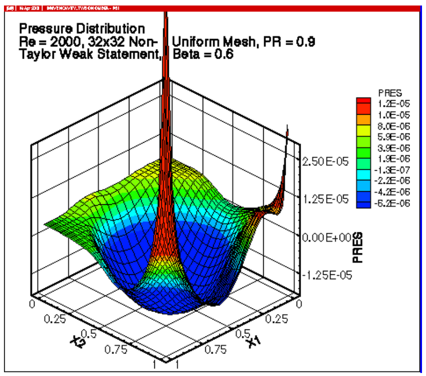


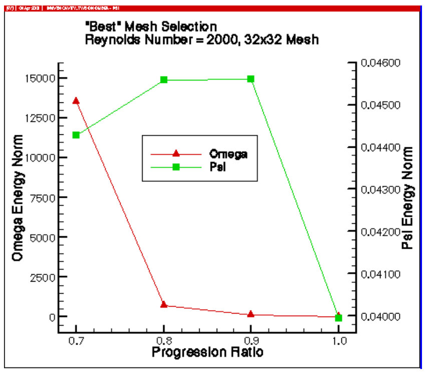
SVNS.16 TWS^h (β) (ω^h, ψ^h) INS Monotone Optimal Solution

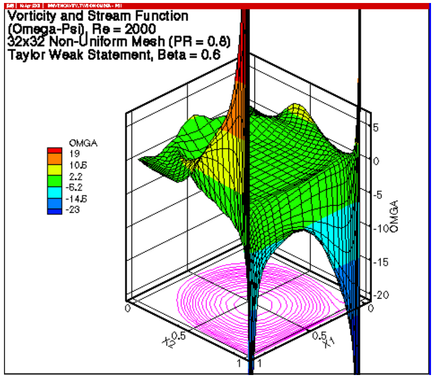
Employ *r* mesh refinement with β = 0.6 ⇒ optimal mesh solution

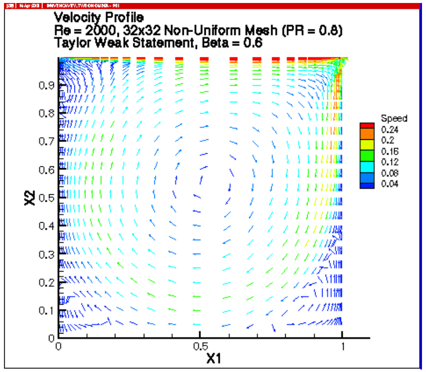


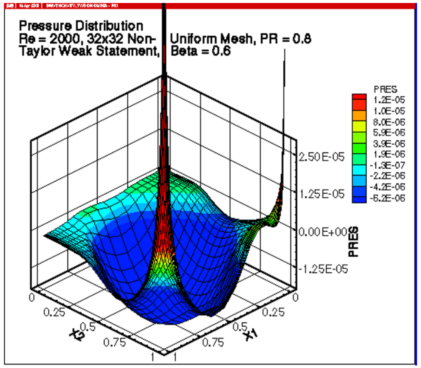


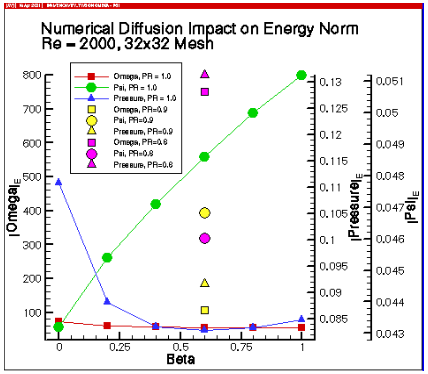








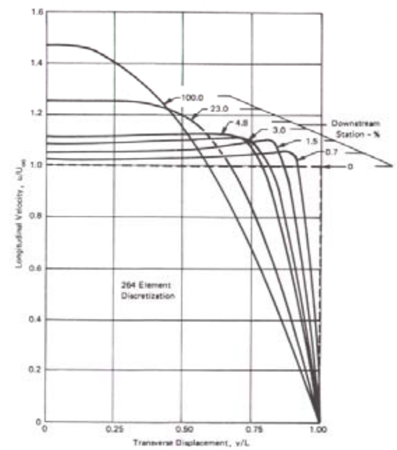


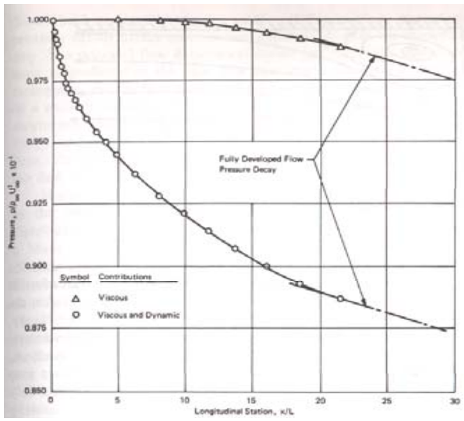


SVNS.17 GWS^h (ω^h, ψ^h) Algorithm, Benchmark, Validation

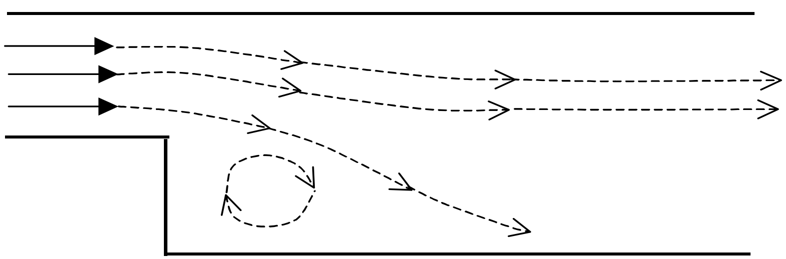
Benchmark problem, n = 2, {$N_1(\zeta)$}:

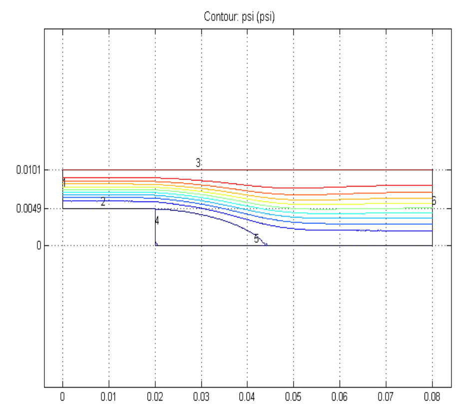
- developing duct flow

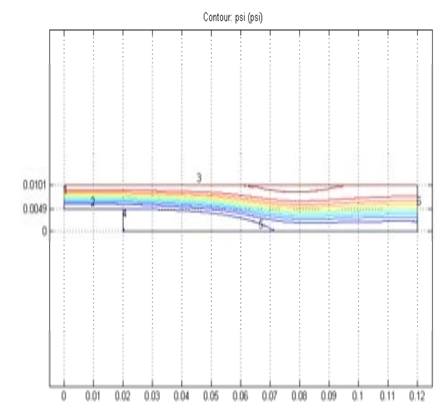


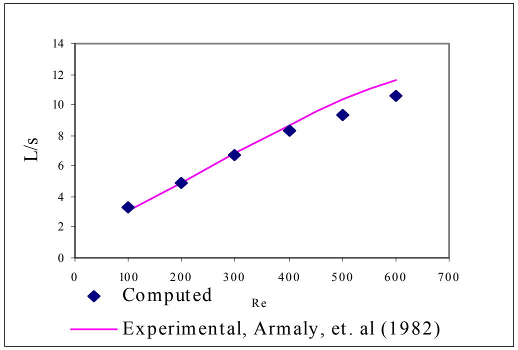


- step-wall diffuser

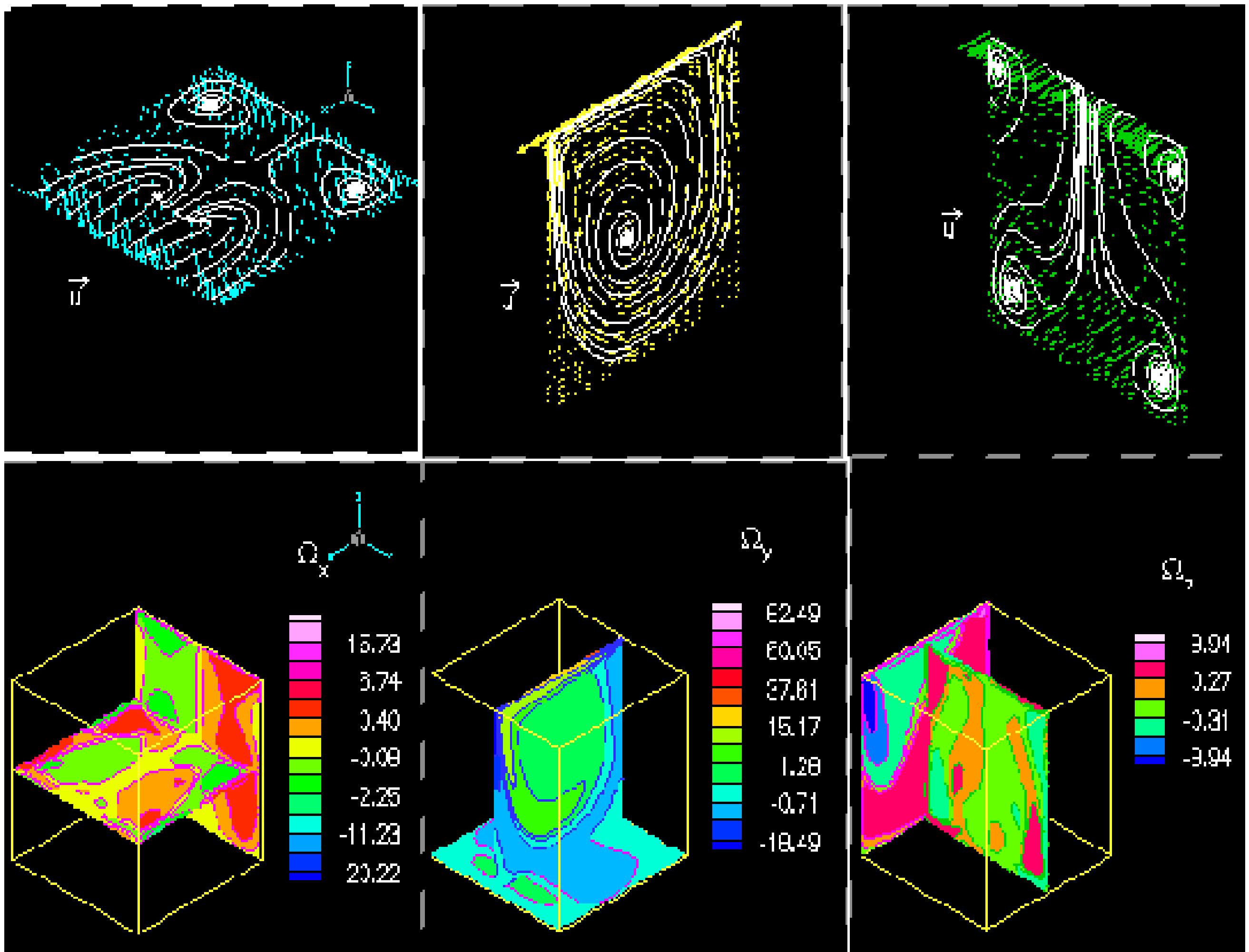








SVNS.18 GWS^h (Ω^h, u^h) Benchmark, n = 3, $\{N_1^+(\eta)\}$



SVNS.19 Thermal INS, (ω^h, ψ^h) Natural-Mixed Convection

INS applicable to buoyant flow simulation via Boussinesq

$n = 2,\ q = \{\omega, \psi, \Theta\}$

$$\begin{aligned}
\mathrm{D}\mathbf{P}:&\quad \mathrm{L}(\omega) = \partial\omega/\partial t + \nabla\times\psi\hat{\mathbf{k}}\cdot\nabla\omega - \mathrm{Re}^{-1}\nabla^2\omega + \mathrm{Gr}\,\mathrm{Re}^{-2}\,\nabla\times\Theta\hat{\mathbf{g}}\cdot\hat{\mathbf{k}} = 0\\
\mathrm{D}M:&\quad \mathrm{L}(\psi) = -\nabla^2\psi - \omega = 0\\
\mathrm{D}E:&\quad \mathrm{L}(\Theta) = \partial\Theta/\partial t + \nabla\times\psi\hat{\mathbf{k}}\cdot\nabla\Theta - \mathrm{Pe}^{-1}\nabla^2\Theta = 0
\end{aligned}$$

BCs

$$\begin{aligned}
\partial\Omega_{\mathrm{in}}:&\quad q(y, x_{\mathrm{in}}, t) \Rightarrow \omega_{\mathrm{in}}, \psi_{\mathrm{in}}, \Theta_{\mathrm{in}}\\
\partial\Omega_{\mathrm{out}}:&\quad \hat{\mathbf{n}}\cdot\nabla(\omega,\psi,\Theta) = 0\\
\partial\Omega_{\mathrm{wall}}:&\quad \psi = \psi_w,\ \Theta = \Theta_w\\
&\quad \hat{\mathbf{n}}\cdot\nabla\omega = f_w(\psi,\omega)
\end{aligned}$$

Thermal effect closure, non-D groups

Boussinesq:

$$\begin{aligned}
(\rho/\rho_0)\mathbf{g} &\Rightarrow \mathrm{Gr}\,\mathrm{Re}^{-2}\,\Theta\hat{\mathbf{g}}\\
\beta &\Rightarrow \mathrm{T}_{\mathrm{abs}}^{-1}\\
\Theta &= (T - \mathrm{T}_{\mathrm{min}})/\Delta T
\end{aligned}$$

$$\begin{aligned}
\text{Grashoff} &= \mathrm{Gr} = \rho_0^2\beta g\Delta \mathrm{T} L^2/\mu^2\\
\text{Reynolds} &= \mathrm{Re} = \rho_0 \mathrm{U} L/\mu\\
\text{Prandtl} &= \mathrm{Pr} = \rho_0 c_p \mu/k\\
\text{Peclet} &= \mathrm{Pe} = \mathrm{Re}\,\mathrm{Pr}\\
\text{Rayleigh} &= \mathrm{Ra} = \frac{\beta g k \Delta T}{\rho c_p \mu \mathrm{U}^2} = \mathrm{Gr}\,\mathrm{Pr}
\end{aligned}$$

SVNS.20 Thermal INS, non-Dimensionalization

Natural convection is missing the velocity scale U

potential scales:

viscous balance $\Rightarrow$ $U \equiv \upsilon L^{-1}$

thermal balance $\Rightarrow$ $U \equiv k/\rho c_p L$

work balance $\Rightarrow$ $1/2\rho_0 U^2 \equiv F_{buoy} L = \rho_0 g \beta \Delta T L$

resultant non-D groups become

balance	U	Re	Gr	$GrRe^{-2}$	Pe
viscous	υL^{-1}	1	•	Gr	Pr
thermal	k/κ	Pr^{-1}	•	PrRa	1
work	$\sqrt{g\beta L \Delta T}$	$\sqrt{Gr}$	•	1	$Pr\sqrt{Gr}$
work	•	$\sqrt{Ra/Pr}$	•	1	$\sqrt{Ra\,Pr}$

notes: Ra is quite often the reference state of choice
non-work scale places large (!) coefficient on **DP** source

SVNS.21 Thermal INS GWS^h + θTS for (ω^h,ψ^h,Θ^h)

GWS^h + θTS ⇒ Newton statement

$$\Rightarrow [\mathrm{JAC}]\{\delta Q\}^{p+1} = -\{\mathrm{F}Q\}^p, \text{ on } \{Q(t)\} = \{\mathrm{OMG, PSI, TEM}\}^T$$

$$[\mathrm{JAC}] = \mathrm{S}_e \begin{bmatrix} \mathrm{J\Omega\Omega}, & \mathrm{J\Omega\Psi}, & \mathrm{J\Omega T} \\ \mathrm{J\Psi\Omega}, & \mathrm{J\Psi\Psi}, & 0 \\ 0\ , & \mathrm{JT\Psi}, & \mathrm{JTT} \end{bmatrix}_e \cong \begin{bmatrix} \mathrm{J\Omega\Omega}, & \mathrm{J\Omega\Psi} \\ \mathrm{J\Psi\Omega}, & \mathrm{J\Psi\Psi} \end{bmatrix}, [\mathrm{JTT}]_e$$

A quasi-Newton approximation responds to the zeros in [JAC]

solution sequence:

$$[\mathrm{JTT}]\{\delta\mathrm{TEM}\}^{p+1} = -\{\mathrm{FTEM}\}^p$$

$$\text{update}: \{\mathrm{TEM}\}^{p+1} = \{\mathrm{TEM}\}^p + \{\delta\mathrm{TEM}\}^{p+1}$$

$$\begin{bmatrix} \mathrm{J\Omega\Omega}, & \mathrm{J\Omega\Psi} \\ \mathrm{J\Psi\Omega}, & \mathrm{J\Psi\Psi} \end{bmatrix} \begin{Bmatrix} \delta\mathrm{OMG} \\ \delta\mathrm{PSI} \end{Bmatrix}^{p+1} = -\left\{\mathrm{F}(\mathrm{OMG}^p, \mathrm{PSI}^p, \mathrm{TEM}^{p+1})\right\}$$

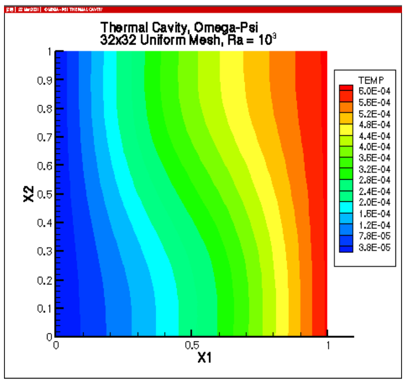
$$\text{update}: \{\mathrm{OMG, PSI}\}^{p+1}$$

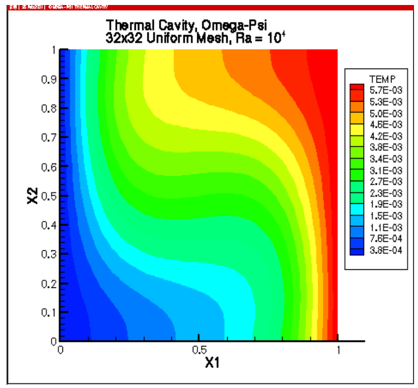
return to [JTT]

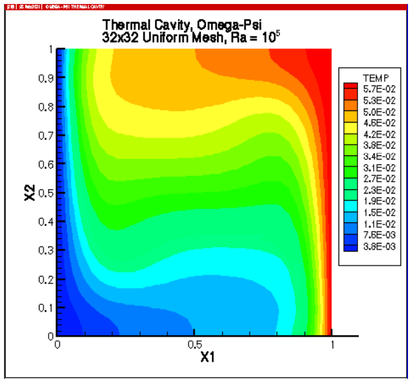
convergence robustness depends on Ra and/or Gr/Re^2

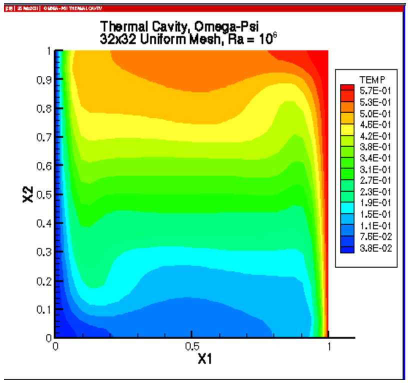
SVNS.22 Thermal INS GWS^h (ω^h, ψ^h, Θ^h) Algorithm Performance

Isotherm distributions, uniform M = 32^2 mesh, $10^3 \leq Ra \leq 10^6$

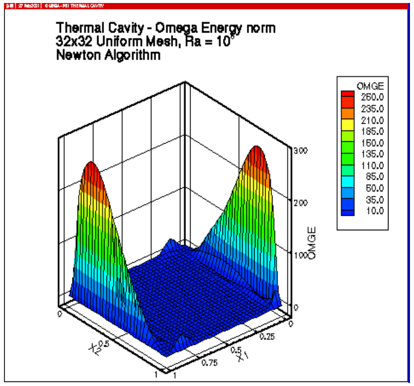


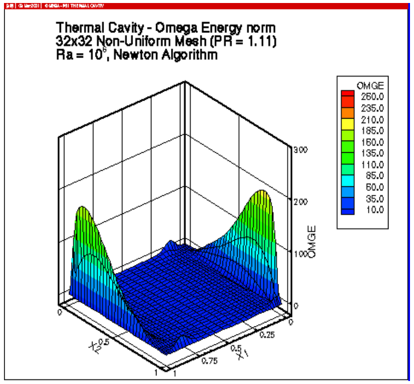


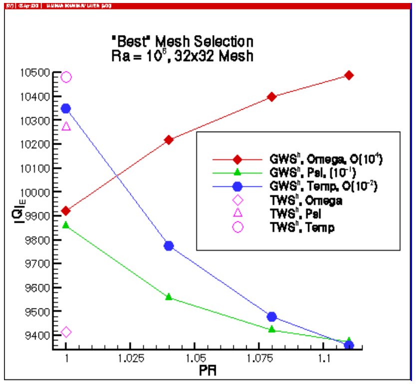


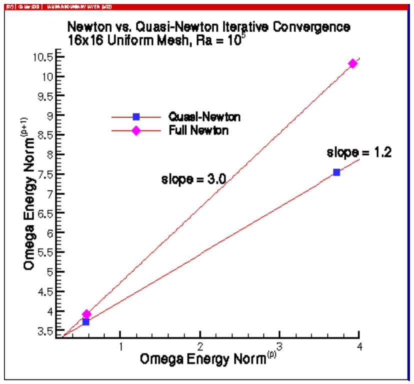


Norm equalization, "best mesh solutions, M = 32^2 meshes, Ra = 10^6









SVNS.23 Thermal INS GWS^h (Ω^h, u^h) Performance, n = 3