CFDLES.1 CFD Algorithm for Large Eddy Simulation

LES NS modified conservation principle PDE systems

$$\overline{\mathrm{D}M} : \nabla \bullet \bar{\mathbf{U}} = 0 = \nabla \bullet \bar{\mathbf{u}}' = \nabla \bullet \mathbf{w}$$

$$\overline{\mathrm{D}\mathbf{P}} : \ \mathrm{L}(\bar{\mathbf{U}}) = \partial \bar{\mathbf{U}} / \partial t + \bar{\mathbf{U}} \bullet \nabla \bar{\mathbf{U}} + \frac{1}{\rho_0} \nabla \bar{\rho} - \upsilon \nabla^2 \bar{\mathbf{U}} + \nabla (\overline{u_i' u_j'}) = 0$$

$$\mathrm{L}(\mathbf{w}) = \mathbf{w}_t + \mathbf{w} \bullet \nabla \mathbf{w} + \nabla r - \nabla \bullet \big((2\upsilon + \upsilon_{\mathrm{T}}) \mathrm{D}(\mathbf{w}) \big)$$

$$+ \nabla \bullet \frac{\delta^2}{2\gamma} \left[\mathrm{A}^{-1} (\nabla \mathbf{w} \nabla \mathbf{w}^T) \right] - \mathbf{f} = 0$$

Closure model definitions, John notation

$$\text{laminar NS} : \upsilon_T = 0, \ \ A = 0$$

$$\text{Smagorinsky} : \upsilon_T = c_s \delta^2 \ \| \mathrm{D}(\mathbf{w}) \|_F , \ A = 0$$

$$\text{Taylor LES} : \upsilon_T = c_s \delta^2 \ \| \mathrm{D}(\mathbf{w}) \|_F , \ A = \lceil \mathrm{I} \rfloor$$

$$\text{rational LES with auxiliary problem}, \ A^{-1} = \left[\mathrm{I} - \frac{\delta^2}{4\gamma} \nabla^2 \right]$$

$$\text{Smagorinsky} : \upsilon_T = c_s \delta^2 \ \| \mathrm{D}(\mathbf{w}) \|_F$$

$$\text{Iliescu-Layton} : \upsilon_T = c_s \delta^2 \ \| \mathbf{w} - A\mathbf{w} \|_2$$

$$\text{Grubert-Baker} : \upsilon_T = \left(\mathrm{Re} h^2(\delta)/12 \right) \mathbf{w}\mathbf{w}^T$$

CFDLES.2 CFD Algorithm for Large Eddy Simulation

GWS^N + θTS CFD algorithm for incompressible LES Navier-Stokes

$$\overline{\mathrm{D}(\cdot)}:\ \mathrm{L}^m(q)=\frac{\partial q}{\partial t}+\frac{\partial}{\partial x_j}\left(f_j-f_j^v\right)-s=0,\ on\ \ \Omega\times t\subset\Re^n\times\Re^1$$

$$\mathrm{L}(q_A)\ =\ -\nabla^2 q_A-s(q,q_A)=0$$

$$\overline{\mathrm{INS}}:q\ =\ \left\{u,v,w,T,(\overline{u_i'u_j'}),..;\phi,P\right\},\ \text{and}\ \nabla\cdot\mathbf{u}=0$$

$$f_j=f_j(u_jq,\mathrm{Eu}p\delta_{ij})$$

$$f_j^v=f_j^v(q,\mathrm{Re},\mathrm{Pr},A,\upsilon_T,\beta)$$

$$s=s(q,\ \mathrm{Gr},\ \mathrm{Re})$$

$$\text{approximation}:q(\mathbf{x},t)\cong q^N(\mathbf{x},t)\equiv\sum_{\alpha=1}^{N}\Psi_\alpha(\mathbf{x})Q_\alpha(t)$$

$$\mathrm{GWS}^N:\int_\Omega\Psi_\beta\mathrm{L}^m(q^N)\mathrm{d}\tau\Rightarrow[\mathrm{MASS}]\frac{\mathrm{d}\{Q\}}{\mathrm{d}t}+\{\mathrm{RES}\}=\{0\}$$

$$\mathrm{GWS}^N+\theta\mathrm{TS}:\{\mathrm{F}Q\}=[\mathrm{MASS}]\{\Delta Q\}+\Delta t\{\mathrm{RES}(Q,Q_A,\theta)\}\equiv\{0\}$$

$$\text{solution}:[\mathrm{JAC}]\{\delta Q\}^{p+1}=-\{\mathrm{F}Q\}^p$$

$$\{Q\}_{n+1}=\{Q\}_n+\sum_{\alpha=0}^{p}\{\delta Q\}^{\alpha+1}$$

CFDLES.3 CFD Algorithm for Large Eddy Simulation

Dissipative flux vector definition for rational LES $\overline{\mathrm{DP}}$

$$f_{ij}^{\upsilon} = 2\upsilon S_{ij} + \upsilon_T S_{ij} - (\delta^2/2\gamma) A(X_{ij}) + (\beta \mathrm{Re} h^2/12) u_j u_k S_{ik}$$

$$S_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$$

$$\upsilon_T = \mathrm{c_s}\delta^2 \left\| \mathsf{D}(\mathbf{u}) \right\|_F = \mathrm{c_s}\delta^2 \left[S_{ij} S_{ij} \right]^{1/2}, \text{ Smagorinsky}$$

$$\upsilon_T = \mathrm{c_s}\delta^2 \left\| \mathbf{u} - A^{-1}\mathbf{u}) \right\|_2 = \mathrm{c_s}\delta^2 \left[\int_{\Omega} \left[u_j - A^{-1}(u_j) \right]^2 \mathrm{d}\tau \right]^{1/2}, \text{ Iliescu-Layton}$$

Auxiliary variable q_A PDEs

$$\overline{\mathrm{DM}} : \quad \mathsf{L}(\varphi) = -\nabla^2\varphi - \nabla \bullet \mathbf{u} = 0$$

$$\mathrm{r\ LES}A : \mathsf{L}(\mathrm{X}_{ij}) = -\frac{\delta^2}{4\gamma}\nabla^2 \mathrm{X}_{ij} + \mathrm{X}_{ij} - \nabla\mathbf{u}\nabla\mathbf{u}^T \Rightarrow \nabla^2 u_i u_j = 0$$

$$\nabla \bullet \overline{\mathbf{DP}} : \quad \mathsf{L}(P) = -\nabla^2 P - \frac{\partial^2}{\partial x_i \partial x_j}\left[u_i u_j + \overline{u_i' u_j'} + f(\mathrm{Gr}, T, A, \beta) \right] = 0$$

$$\mathrm{BCs} : \quad l(q_A) = \hat{\mathbf{n}} \bullet \nabla q_A + g(q, q_A, A, \mathrm{Re}) = 0$$

CFDLES.4 CFD Algorithm for Large Eddy Simulation

LES CFD models definitely run on the balance of dissipative flux vectors

it is fundamental to quantitatively assess contributions

GWS^h for flux vector computation

$$\mathrm{L}(S_{ij}) = S_{ij} - \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] = 0$$

$$\mathrm{GWS}(S_{ij}) = \int_{\Omega} \Psi_\beta(\mathbf{x})\,\mathrm{L}(S_{ij}^N)\, d\tau = 0 = \mathrm{S}_e\{\mathrm{WS}\}_e$$

$$\{\mathrm{WS}(S_{ij}^h)\} = \frac{1}{2}\int_{\Omega_e}\{N\}\left[2\{N\}^T\{\mathrm{S}IJ\}_e - \frac{\partial\{N\}^T}{\partial x_j}\{\mathrm{U}I\}_e - \frac{\partial\{N\}^T}{\partial x_i}\{\mathrm{U}J\}_e\right] d\tau$$

$$= \frac{1}{2}\int_{\Omega_e}\{N\}\left[2\{N\}^T\{\mathrm{S}IJ\}_e - \frac{\partial\{N\}^T}{\partial \eta_k}\frac{\partial \eta_k}{\partial x_j}\{\mathrm{U}I\}_e - \frac{\partial\{N\}^T}{\partial \eta_k}\frac{\partial \eta_k}{\partial x_i}\{\mathrm{U}J\}_e\right] d\tau$$

$$\text{template: } \{\mathrm{FS}IJ\} = (\ )(\ )\{\ \}(\ ;1)[\mathrm{M200}]\{\mathrm{S}IJ\} + (-0.5)(\ )\{\ \}(\mathrm{E}KJ;0)[\mathrm{M20}K]\{\mathrm{U}I\}$$
$$+ (-0.5)(\ )\{\ \}(\mathrm{E}KI;0)[\mathrm{M20}K]\{\mathrm{U}J\}$$

$$[\mathrm{JS}IJ]_e = (\ )(\ )\{\ \}(\ ;1)[\mathrm{M200}]\left[\ \right]$$

CFDLES.5 CFD Algorithm for Large Eddy Simulation

Use of mean flow strain rate nodal distributions

υ, υ_T and β terms are direct to compute

$$\mathrm{L}(\mathrm{FB}IJ) = \mathrm{F}\beta_{ij} - (\beta\mathrm{Re}/12)h^2 u_j u_k S_{ik} \equiv 0$$

$$\mathrm{GWS}(\mathrm{F}\beta_{ij}) = \ldots = \mathrm{S}_e\{\mathrm{WS}\}_e = 0$$

$$\text{template}: \{\mathrm{FB}IJ\} = (\ )(\ )(\ )(\ ;1)[\mathrm{M200}]\{\mathrm{FB}IJ\}$$
$$+(-\beta, \mathrm{Re}/12)(h^2)(\mathrm{U}J, \mathrm{U}K)(\ ;1)[\mathrm{M3000}]\{\mathrm{S}IK\}$$

$$[\mathrm{JB}IJ]_e = (\ )(\ )\{\ \}(\ ;1)[\mathrm{M200}]$$

LES auxiliary problem

$$\mathrm{L}(X_{ij}) = -\frac{\delta^2}{4\gamma}\nabla^2 X_{ij} + X_{ij} - \nabla^2 u_i u_j = 0$$

$$\text{template}: \{\mathrm{FX}IJ\}_e = (\ )(\ )\{\ \}(\ ;1)[\mathrm{M200}]\{\mathrm{X}IJ\}$$
$$+(\frac{\delta^2}{4\gamma})(\ )\{\ \}(\mathrm{E}KP, \mathrm{E}LP; -1)[\mathrm{M2}KL]\{\mathrm{X}IJ\}$$
$$+(\ )(\ )\{\mathrm{U}J\}(\mathrm{E}KP, \mathrm{E}LP; -1)[\mathrm{M30}KL]\{\mathrm{U}I\}$$
$$+(\ )(\ )\{\mathrm{U}I\}(\mathrm{E}KP, \mathrm{E}LP; -1)[\mathrm{M30}KL]\ [UJ]$$

CFDLES.6 CFD Algorithm for Large Eddy Simulation

Summary, weak form CFD algorithms for LES NS models

all ingredients are identified via the weak solution process

rational LES algorithm does depend on filter width

a quasi-Newton approach is undoubtedly required

auxilliary problem solution retarded iteratively

phi solution should not be retarded

John + co-workers successfully employ a multi-grid approach

using inf-sup compatible FE bases, settled on $Q_2 / P_1^{\mathrm{disc}}$

stability requires implementation of $\overline{u'_i u'_j}$ model

Taylor LES is verified unstable

rational LES is verified stable only for $c_s > 0$

auxilliary problem process preferable to convolution

much remains to be explored

stability, balance of dissipative mechanisms

$\overline{u'_i u'_j}$ model of $O(\delta^4)$

efficient linear algebra

accuracy/convergence

verification,validation