LES.1 Large Eddy Simulation Model for Turbulent Flow

Direct numerical simulation, reprise (Pope, 2000)

DNS involves solving unsteady NS resolving *all* scales of motion

homogeneous turbulence $\Rightarrow$ periodic BCs
pseudo-spectral CFD
$\boldsymbol{\kappa}$ – space solution process
Fourier transforms

in homogeneous turbulence $\Rightarrow$ non-periodic BCs, near-wall resolution
Fourier transforms not useable

Homogeneous turbulence, pseudo-spectral CFD

Ω is a cube of measure l

$$\mathbf{u}(\mathbf{x},t) \equiv \sum_{\boldsymbol{\kappa}} e^{i\boldsymbol{\kappa}\bullet\mathbf{x}}\hat{\mathbf{u}}(\boldsymbol{\kappa},t)$$

wave number resolution for N^3 modes

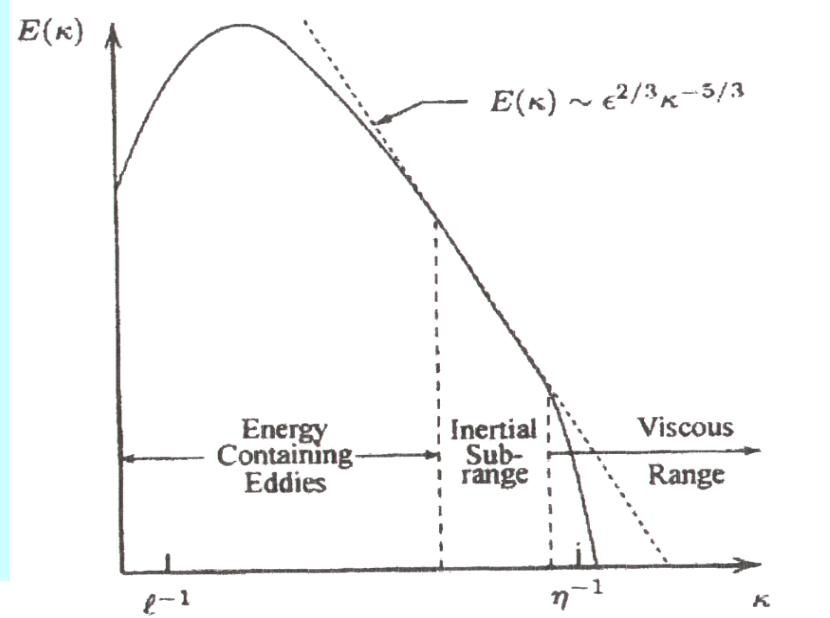
$$\kappa_0 = 2\pi/l \, , \kappa_{max} = \pi N/l$$

equivalent physical space resolution

$$\Delta x = l/N = \pi/\kappa_{max}$$

pseudo-spectral transports non-linear NS terms
physical $\Leftrightarrow$ wave number space

99% effort goes to dissipation range resolution



LES.2 Large Eddy Simulation Model for Turbulent Flow

Homogeneous turbulence, pseudo-spectral resolution requirements

smallest scale of motion is Kolmogrov scale η

adequate resolution : $\kappa_{max}\eta \geq 1.5 \Leftrightarrow \Delta x/\eta = \pi/1.5 \sim 2$

largest scale l must be sufficient to contain energy containing motions

lower limit is l = 8 integral length scales L_{11}

integral length scale from two-point, one-time autocovariance, $i = 1 = j$

$$\langle u_i(\mathbf{x},t)\rangle\langle u_j(\mathbf{x}+\mathbf{r},t)\rangle \Rightarrow L_{11}(\mathbf{x},t) = \frac{1}{R_{11}(0,\mathbf{x},t)}\int_{-\infty}^{\infty} R_{11}(r\hat{e}_1,\mathbf{x},t)\,\mathrm{d}\mathbf{r}$$

combination leads to : $N \sim 1.6(l/\eta) = 1.6\mathrm{Re}_l^{3/4}$

$$N^3 \sim 4.4\mathrm{Re}_l^{9/4}$$

integration time step : Courant = $k^{1/2}\Delta t / \Delta x \sim 0.05$

time integration duration of order 4 turbulence timescales $\tau = k/\varepsilon$

$$M = \frac{4\tau}{\Delta t} = 80\frac{l}{\Delta x} = \frac{120}{\pi}\frac{l}{\eta}$$

computing time requirement at 1 gflop rate (days)

$$T_{\mathrm{days}} = \frac{10^3 N^3 M}{10^9 \text{ x } 60 \text{ x } 60 \text{ x } 24} \sim \left(\frac{\mathrm{Re}_l}{800}\right)^3 \sim \left(\frac{\mathrm{Re}_l}{70}\right)^6$$

$\sim$ 5000 years for $\mathrm{Re}_l \sim 10^5$

LES.3 Large Eddy Simulation for Turbulent Flow

Categorization by motion resolution for turbulent NS

Model	Acronym	Resolution
Direct numerical simulation	DNS	Turbulent motions of all scales are fully resolved
Large-eddy simulation with near-wall resolution	LES-NWR	The filter and grid are sufficiently fine to resolve 80% of the energy everywhere
Large-eddy simulation with near-wall modelling	LES-NWM	The filter and grid are sufficiently fine to resolve 80% of the energy remote from the wall, but not in the near-wall region
Very-large-eddy simulation	VLES	The filter and grid are too coarse to resolve 80% of the energy
Detached eddy simulation	DES	A variation on LES-NNM

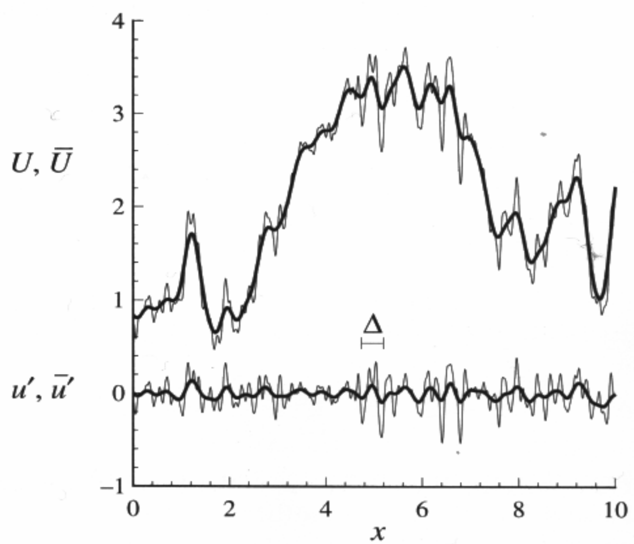
LES for random scalar function $U\,(x,\,t)$

$$\bar{U}(x,t) \equiv \int_{-\infty}^{\infty} G(r)\, U(x\text{-}r,t)\, \mathrm{d}r$$

which is the convolution integral

$$u'(x,t) \equiv U(x,t) - \bar{U}(x,t)$$

$$\overline{u'(x,t)} \equiv \int_{-\infty}^{\infty} G(r)\, u'(x\text{-}r,t)\, \mathrm{d}r$$



for G the Gaussian with $\Delta = 0.35$

LES.4 Large Eddy Simulation Model for Turbulent Flow

Large eddy simulation- the compromise between RaNS and DNS

process: define mean velocity

$$\bar{U}_i(\mathbf{x},t) \equiv \int_\Delta G(\mathbf{x}\text{-}\xi;\Delta)\, U_i(\xi,t)\ \mathrm{d}^3\xi$$

define the fluctuation

$$U_i(\mathbf{x},t) \equiv \bar{U}_i(\mathbf{x},t) + u_i'(\mathbf{x},t)$$

filtered NS equations produces residual stress tensor

$$\overline{U_i U_j} \text{ and } \overline{u_i'(\mathbf{x},t)} \neq 0$$

resolve filtered NS residual stress

$$\overline{U_i U_j} \equiv \bar{U}_i \bar{U}_j + L_{ij} + C_{ij} + R_{ij}$$

evaluate Leonard and cross-term stresses from filter $G(\bullet)$

$$L_{ij} \equiv \overline{\bar{U}_i \bar{U}_j} - \bar{U}_i \bar{U}_j$$

$$C_{ij} \equiv \overline{\bar{U}_i u_j'} + \overline{\bar{U}_j u_i'}$$

select sub-grid scale (SGS) closure model for R_{ij}

$$R_{ij} \equiv \overline{u_i' u_j'}$$

solve filtered NS equations for $\bar{U}_i(\mathbf{x},t)$

LES.5 Large Eddy Simulation Model for Turbulent Flow

Fourier transforms reprise (Pope, 2000, App.D)

given $f(t)$, the Fourier transform is

$$g(\omega) \equiv \mathcal{F}\{f(t)\} \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt$$

inverse transform : $f(t) \equiv \mathcal{F}^{-1}\{g(\omega)\} \equiv \int_{-\infty}^{\infty} g(\omega)\, e^{-i\omega t} d\omega$

Useful Fourier transform pairs, operations

derivatives : $\mathcal{F}\left\{\frac{d^n f(t)}{dt^n}\right\} = (i\omega)^n g(\omega)$

$$\mathcal{F}^{-1}\left\{\frac{d^n f(t)}{dt^n}\right\} = (-it)^n f(t)$$

cosine transform : for $f(t)$ real and even ($\equiv f(-t)$)

$$g(\omega) = \frac{1}{\pi}\int_0^{\infty} f(t)\ \cos\omega t\ dt$$

$$f(t) = 2\int_0^{\infty} g(\omega)\cos\omega t\ d\omega$$

$f(t)$	$g(\omega)$
1	$\delta(\omega)$
$\delta(t-a)$	$\frac{1}{2\pi}e^{-i\omega a}$
$\delta^{(n)}(t-a)$	$\frac{(i\omega)^n}{2\pi}e^{-i\omega a}$
$e^{-b\lvert t\rvert}$	$\frac{b}{\pi(b^2+\omega^2)}$
$\frac{1}{b\sqrt{2\pi}}e^{-t^2/(2b^2)}$	$\frac{1}{2\pi}e^{-b^2\omega^2/2}$
$H(b-\lvert t\rvert)$	$\frac{\sin(b\omega)}{\pi\omega}$
$(b^2+t^2)^{-(\nu+1/2)}$	$\frac{2\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})}\left(\frac{\lvert\omega\rvert}{2b}\right)^{\nu} K_\nu\left(\frac{\lvert\omega\rvert}{b}\right)$

LES.6 Large Eddy Simulation Model for Turbulent Flow

Useful Fourier transform pairs, operations, concluded

delta function : $\mathsf{F}\{\delta(t\text{-}a)\} = \dfrac{1}{2\pi}\mathrm{e}^{-i\omega a}$

$$\delta(t\text{-}a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-i\omega(t-a)}\mathrm{d}\omega \text{ for } \mathrm{g}(\omega) = \frac{1}{2\pi}\mathrm{e}^{-i\omega a}$$

$$\int_{-\infty}^{\infty} G(t)\ \delta(t\text{-}a)\ \mathrm{d}t = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi}G(t)\mathrm{e}^{-i\omega(t-a)}\mathrm{d}t\ \mathrm{d}\omega = G(a)$$

convolution : $h(t) \equiv \displaystyle\int_{-\infty}^{\infty} f_1(t-s)f_2(s)\ \mathrm{d}s$

$$\mathsf{F}\{h(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-i\omega t}\int_{-\infty}^{\infty} f_1(t-s)f_2(s)\ \mathrm{d}s\mathrm{d}t$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-i\omega(r+s)} f_1(r)f_2(s)\ \mathrm{d}s\mathrm{d}t$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-i\omega r} f_1(r)\ \mathrm{d}r\int_{-\infty}^{\infty} \mathrm{e}^{-i\omega s} f_2(s)\ \mathrm{d}s$$

$$= 2\pi\ \mathsf{F}\{f_1(t)\}\mathsf{F}\{f_2(t)\}$$

LES.7 Large Eddy Simulation of Turbulent Fluid Dynamics

Spectral representation, filtered variables in wavenumber space

Fourier transform for U : $\hat{U}(\kappa) \equiv \mathsf{F}\{U(x)\}$

for filtered U : $\hat{\overline{U}}(\kappa) \equiv \mathsf{F}\{\overline{U}(x)\} = \hat{G}(\kappa)\hat{U}(\kappa)$

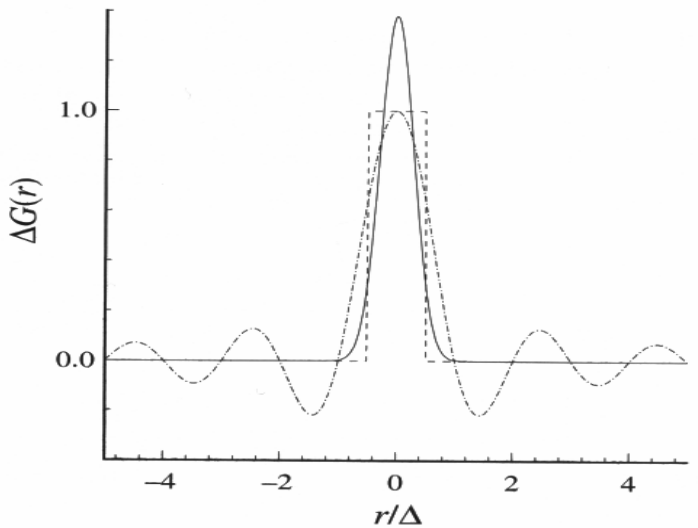
transform function : $\hat{G}(\kappa) \equiv \int_{-\infty}^{\infty} G(r)\mathrm{e}^{-\mathrm{i}\kappa r}\mathrm{d}r = 2\pi\mathsf{F}\{G(r)\}$

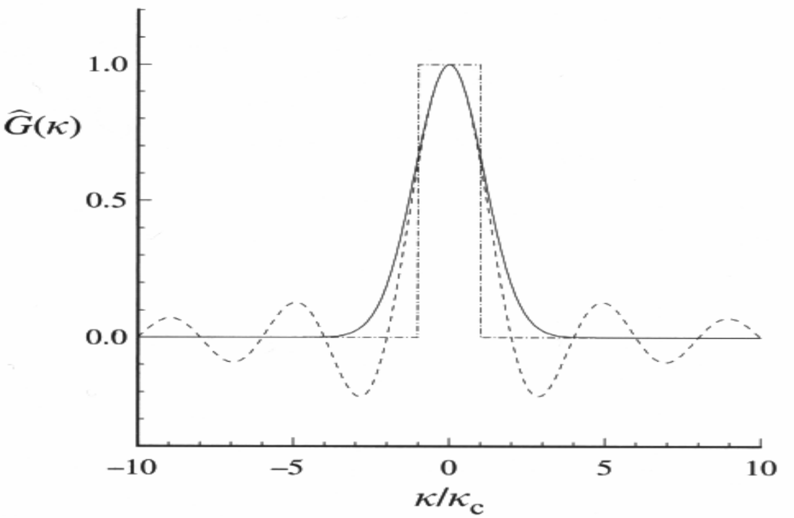
Commentary on filters $G(r)$, hence filter transfer functions $\hat{G}(\kappa)$

box : compact (Heaviside) in r , diffuse in κ

Gaussian : compact in both r and κ

sharp spectral : diffuse in r, compact (Heaviside) in κ

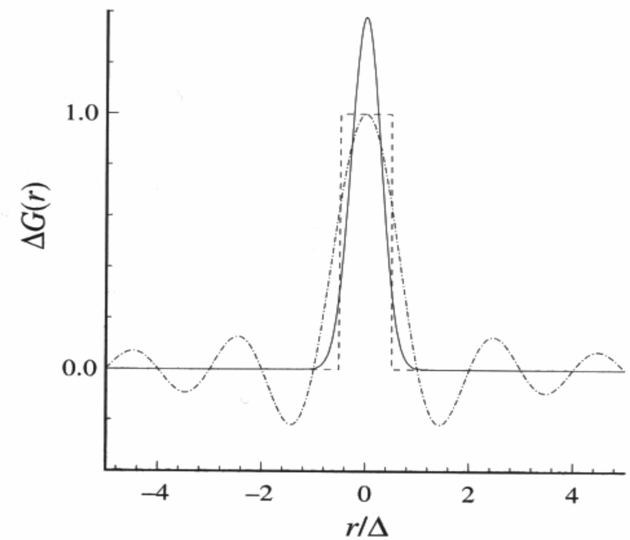


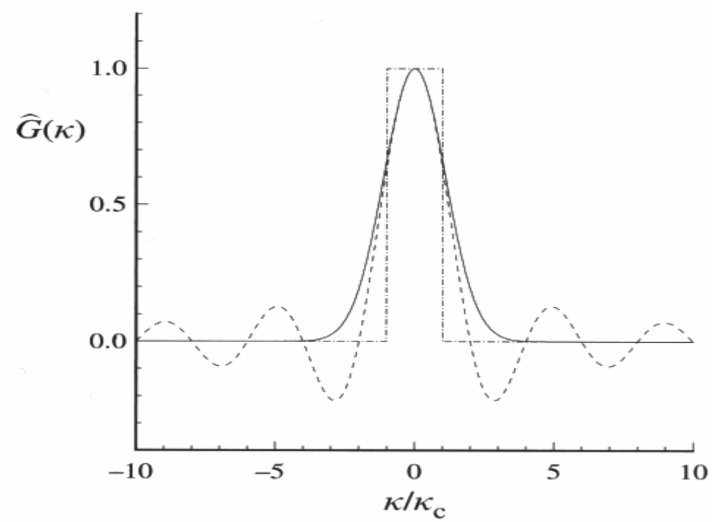


LES.8 Large Eddy Simulation Model for Turbulent Flow

Filter and transfer functions of use in LES

Name	Filter function	Transfer function
General	$G(r)$	$\widehat{G}(\kappa) \equiv \int_{-\infty}^{\infty} e^{i\kappa r} G(r)\, dr$
Box	$\frac{1}{\Delta} H(\frac{1}{2}\Delta - \lvert r \rvert)$	$\frac{\sin(\frac{1}{2}\kappa\Delta)}{\frac{1}{2}\kappa\Delta}$
Gaussian	$\left(\frac{6}{\pi\Delta^2}\right)^{1/2} \exp\left(-\frac{6r^2}{\Delta^2}\right)$	$\exp\left(-\frac{\kappa^2\Delta^2}{24}\right)$
Sharp spectral	$\frac{\sin(\pi r/\Delta)}{\pi r}$	$H(\kappa_c - \lvert\kappa\rvert)$, $\kappa_c \equiv \pi/\Delta$
Cauchy	$\frac{a}{\pi\Delta[(r/\Delta)^2 + a^2]}$, $a = \frac{\pi}{24}$	$\exp(-a\Delta\lvert\kappa\rvert)$
Pao		$\exp\left(-\frac{\pi^{2/3}}{24}(\Delta\lvert\kappa\rvert)^{4/3}\right)$





dashed — box filter
solid — Gaussian filter
dot-dash — sharp spectral filter
κ_c = filter cutoff wave number $= \pi/\Delta$

LES.9 Large Eddy Simulation Model for Turbulent Flow

Filtering effect on energy spectrum, statistically homogeneous $U(x)$

$$\text{resolution}: \quad U(x) \equiv \langle U(x) \rangle + u(x)$$

$$\text{autocovariance}: \quad R(r) \equiv \langle u(x+r) \rangle \langle u(x) \rangle$$

$$\text{energy spectrum}: E_{11}(\kappa) \equiv 2\mathrm{F}\{R\} = \frac{1}{\pi} \int_{-\infty}^{\infty} R(r) e^{-i\kappa r} \mathrm{d}r$$

$$\text{filtered covariance}: \quad \bar{R}(r) \equiv \langle \bar{u}(x+r) \rangle \langle \bar{u}(x) \rangle$$

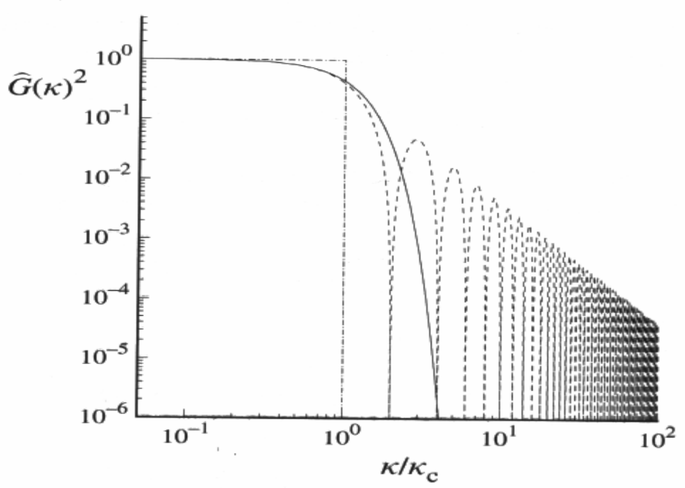
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(y) G(z) R(r+z-y) \mathrm{d}y\mathrm{d}z$$

$$\text{filtered spectrum}: \bar{E}_{11}(\kappa) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \bar{R}(r) e^{-i\kappa r} \mathrm{d}r$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(y) e^{-i\kappa y} G(z) e^{-i\kappa z} R(r+z-y) e^{-i\kappa(r+z-y)} \mathrm{d}y\mathrm{d}z\mathrm{d}r$$

$$= \hat{G}(\kappa)\, \hat{G}^{*}(\kappa) E_{11}(\kappa)$$

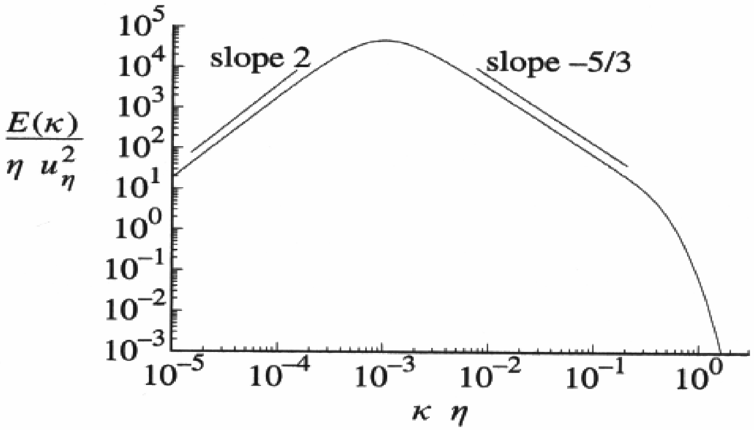
$$= \left| \hat{G}(\kappa) \right|^{2} E_{11}(\kappa)$$



box(---), Gaussian(—), sharp spectral(— •)

LES.10 Large Eddy Simulation Model for Turbulent Flow

Filtering effect on energy spectrum, statistically homogeneous $U(\mathrm{x})$, concluded



a model spectrum : $E(\kappa) \equiv C\varepsilon^{2/3}\kappa^{-5/3} f_L(\kappa L) f_\eta(\kappa\eta)$

$$f_L \equiv \left[\kappa L / [(\kappa L)^2 + c_L]^{1/2}\right]^{5/3+2}$$

$$\Rightarrow E(\kappa) \sim \kappa^2 \text{ for small } \kappa L$$

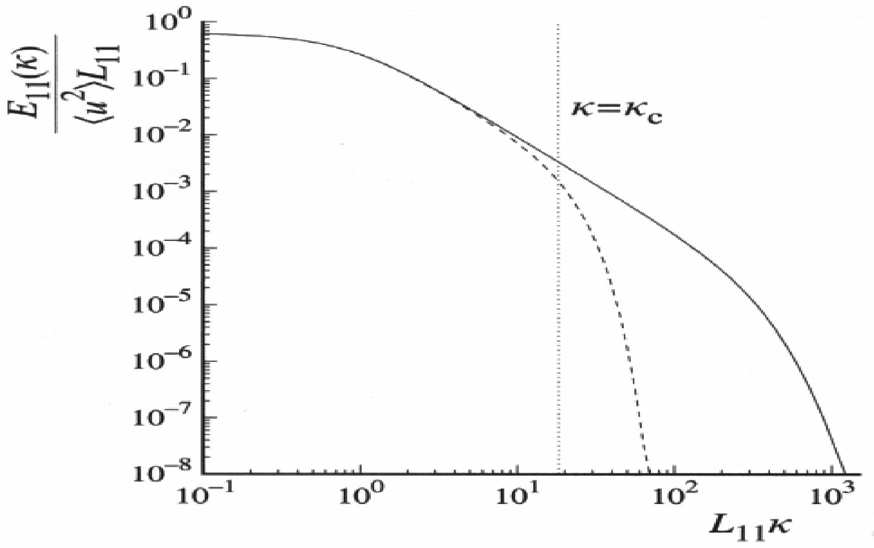
$$f_\eta \sim \exp(-\beta\kappa\eta) \text{ for } 2.1 \le \beta(\kappa\eta) \le 5.2$$

$$\Rightarrow E(\kappa) \sim \text{Kolmogorov for large } \kappa$$

for Taylor Re : $R_\lambda \equiv u'\lambda_g / \upsilon = 500$

$$= (20\mathrm{Re}_L/3)^{1/2}\ ,\ \ \mathrm{Re}_L = k^{1/2}L/\upsilon = k^2/\varepsilon\upsilon$$

Energy spectrum comparisons for Gaussian filter, $\mathrm{Re}_\lambda = 500$



$$\Delta \equiv \frac{1}{6} L_{11} = \frac{1}{\langle u^2 \rangle}\int_0^\infty R(r)\mathrm{d}r \equiv \ell_{EI}$$

$\ell_{EI} \equiv$ demarkation length scale separating energy-containing eddies ($\ell > \ell_{EI}$)from others

$E_{11}(\kappa)$(—) , $\overline{E}_{11}(\kappa)$(- - -)

$\langle \overline{u}^2 \rangle \sim 0.92 \langle u^2 \rangle$, $0.92 \rightarrow 0.80$ in 3-D

LES.11 Large Eddy Simulation Model for Turbulent Flow

Mesh resolution for filtered velocity field, statistically homogeneous $U(x)$

for $\bar{u}(x) \approx \bar{u}^h(x)$ on uniform mesh with h=L/N, $0 \le x \le L$

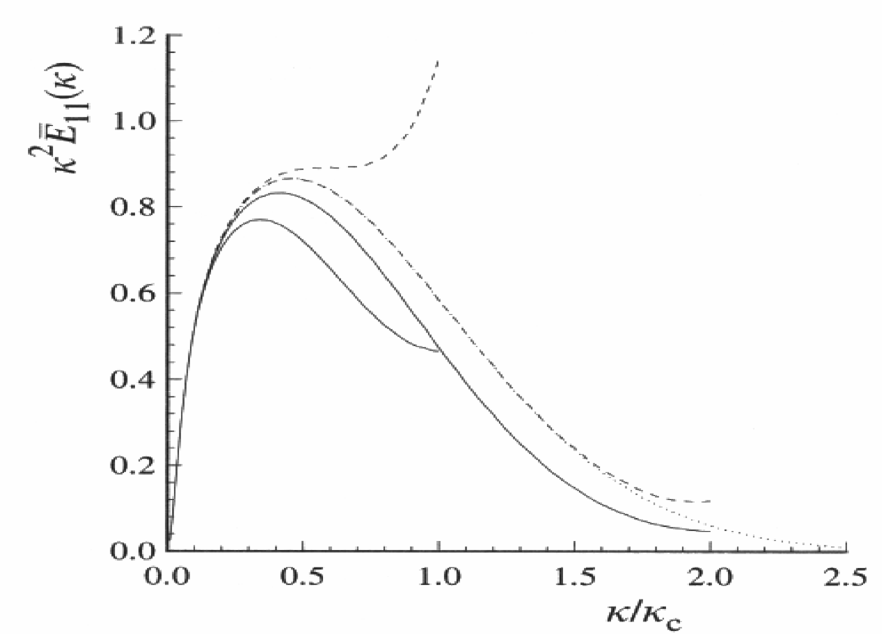
$\kappa_{max} = \pi N_{max} / L$ and $\kappa_{max}\eta \ge 1.5$ for isotropic turbulence

$h_{max} = L / N_{max} = \pi/\kappa_{max}$ is largest adequate mesh

for Gaussian, highest resolved mode is $\kappa_r = \pi/h$, cutoff is $\kappa_c = \pi/\Delta$

resolution : $h/\Delta \sim \kappa_c/\kappa_r = (1/2, 1) \Leftrightarrow$ (good , poor)

Velocity first derivative spectra, $\kappa^2 \bar{E}_{11}(\kappa)$, Gaussian, $R_\lambda = 500$



$d\bar{u}(x)dx$: (.....) model spectrum
$h/\Delta = 1/2$ (98%)
$h/\Delta = 1$ (72%)

$d\bar{u}^I(x)/dx$: (---), $h/\Delta = 1/2$ (ok)
$h/\Delta = 1$ (aliased)

$d\bar{u}^h(x)/dx$: (—), $h/\Delta = 1/2$ (86%)
$h/\Delta = 1$ (60%)
$h/\Delta = 1/4$ (96%)

LES.13 Large Eddy Simulation Model For Turbulent Flow

Filtered NS conservation law system, concluded

$$\mathrm{D}\bar{E} = \bar{U}_j \overline{\mathrm{D}P_j} \Rightarrow \bar{\mathrm{D}}(E_f + k_r)\,,\ E_f \equiv \frac{1}{2}\bar{\boldsymbol{U}} \bullet \bar{\boldsymbol{U}}$$

$$k_r \equiv \frac{1}{2}\overline{\bar{\boldsymbol{U}} \bullet \bar{\boldsymbol{U}}} - \frac{1}{2}\bar{\boldsymbol{U}} \bullet \bar{\boldsymbol{U}} = \frac{1}{2}\tau_{ij}^{\mathrm{R}}$$

$$\overline{\mathrm{D}E} : \frac{\bar{\mathrm{D}}E_f}{\bar{\mathrm{D}}t} - \frac{\partial}{\partial x_j}\left[\bar{U}_j(2\upsilon\bar{S}_{ij} - \tau_{ij}^r - \frac{\bar{p}_r}{\rho_0}\delta_{ij})\right] = -\varepsilon_f - P_r$$

where: $\varepsilon_f \equiv 2\upsilon\bar{S}_{ij}\bar{S}_{ij}$, filtered viscous dissipation

$P_r \equiv -\tau_{ij}^r\bar{S}_{ij}$, filtered $\rightarrow$ residual motion dissipation

$$\text{Strain} : \bar{S}_{ij} \equiv \frac{1}{2}\left(\bar{U}_{i,j} + \bar{U}_{j,i}\right)$$

$$\bar{S} \equiv \left(2\bar{S}_{ij}\bar{S}_{ij}\right)^{1/2} \equiv \text{characteristic filtered strain rate}$$

$$\left\langle \bar{S}^2 \right\rangle \equiv 2\bar{S}_{ij}\bar{S}_{ij} = 2\int_0^\infty \kappa^2 \bar{E}(\kappa)\mathrm{d}\kappa$$

$$= 2\int_0^\infty \kappa^2 \hat{G}(\kappa)^2 E(\kappa)\mathrm{d}\kappa$$

are scales used in SGS closure models

LES.12 Large Eddy Simulation Model for Turbulent Flow

Filtered NS conservation law system

for spatially uniform filters, differentiation and filtering commute

$$\mathrm{D}\bar{M}:\ \overline{\partial U_i/\partial x_i} = \partial \bar{U}_i/\partial x_i = 0 = \partial u_i'/\partial x_i$$

$$\mathrm{D}\bar{P}:\ \frac{\partial \bar{U}_i}{\partial t} + \frac{\partial(\overline{U_j U_i})}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x_i} + \upsilon \frac{\partial^2 \bar{U}_i}{\partial x_j^2}$$

$$\text{anisotropic residual stress tensor}: \tau_{ij}^r \equiv \overline{U_i U_j} - \bar{U}_i \bar{U}_j - \frac{2}{3} k_r \delta_{ij}$$

$$\text{modified pressure}: \bar{p}_r \Rightarrow \bar{p} + \frac{2}{3} k_r$$

$$\overline{\mathbf{DP}}: \bar{\mathrm{D}}(\bar{U}_i)/\bar{\mathrm{D}}t = \frac{1}{\rho_0}\frac{\mathrm{d}p_r}{\mathrm{d}x_i} + \upsilon \frac{\partial^2 \bar{U}_i}{\partial x_j^2} - \partial \tau_{ij}^r/\partial x_j$$

Residual stress tensor resolution, Galilean-invariant form (Germano,1986)

$$\tau_{ij}^{\mathrm{R}} = \tau_{ij}^r + \frac{2}{3} k_r \delta_{ij} \equiv L_{ij}^0 + C_{ij}^0 + R_{ij}^0$$

$$\text{Leonard}: L_{ij}^0 \equiv \overline{\bar{U}_i \bar{U}_j} - \bar{\bar{U}}_i \bar{\bar{U}}_j$$

$$\text{Cross}: C_{ij}^0 \equiv \overline{\bar{U}_i u_j'} + \overline{u_i' \bar{U}_j} - \bar{\bar{U}}_i \bar{u}_j' - \bar{u}_i' \bar{\bar{U}}_j$$

$$\text{SGS Reynolds}: R_{ij}^0 \equiv \overline{u_i' u_j'} - \bar{u}_i' \bar{u}_j'$$

LES.14 Large Eddy Simulation Model for Turbulent Flow

Closure for filtered NS system, the Smagorinsky model

anisotropic residual stress tensor, Boussinesq model

$$\tau_{ij}^{r} \equiv -\upsilon_r 2\bar{S}_{ij}$$

$$\upsilon_r = D(\mathrm{L}^2/t) \equiv \ell_s^2 \bar{S}, \quad \bar{S} \equiv \left(2\bar{S}_{ij}\bar{S}_{ij}\right)^{1/2}$$

Smagorinsky: $\ell_s \equiv \mathrm{C_s}\Delta$

$$P_r \equiv \tau_{ij}^{r}\bar{S}_{ij} = 2\upsilon_r \bar{S}_{ij}\bar{S}_{ij} = \upsilon_r \bar{S}^2 \geq 0$$

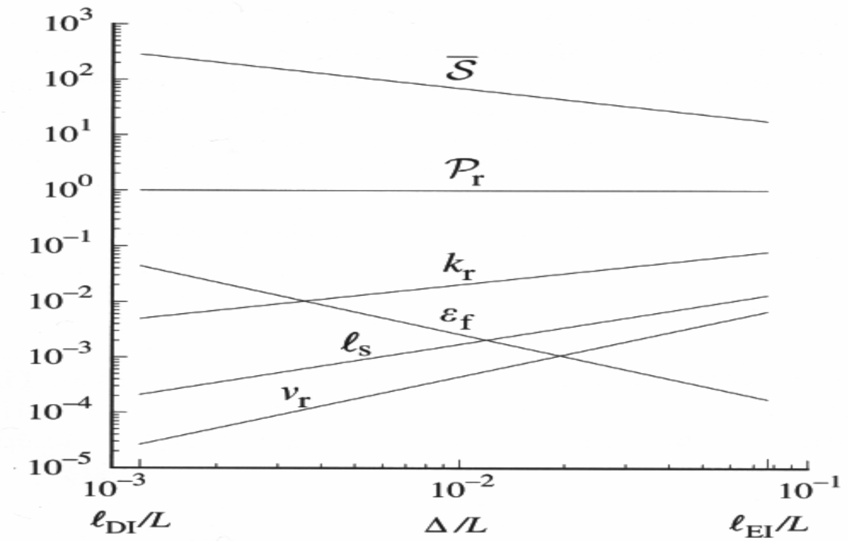
$\therefore$ no back scatter, mean $\Rightarrow$ residual motion only

For high Re turbulence, filter in the inertial subrange, $\ell_{EI} < \Delta < \ell_{DI}$

balance: $\varepsilon = \langle P_r \rangle = \langle \upsilon_r \bar{S}^2 \rangle = \ell_s \langle \bar{S}^3 \rangle$

via Kolmogorov: $\ell_s = \dfrac{\Delta}{(\mathrm{Ca_F})^{3/4}} \left[\dfrac{\langle \bar{S}^3 \rangle}{\langle \bar{S}^2 \rangle} \right]^{-1/2}$

sharp spectral : $\mathrm{C_s} = \dfrac{\ell_s}{\Delta} \approx 0.17$



LES.15 Large Eddy Simulation Model for Turbulent Flow

Smagorinsky filter is uniquely implied in homogeneous isotropic

note : ℓ_s closure model is independent of filter specification

filter transfer function : $\hat{G}(\kappa) = \left[\dfrac{\bar{E}_s(\kappa)}{\bar{E}_{DNS}(\kappa)} \right]$

In inertial subrange, for $\bar{S} \sim \langle \bar{S}^2 \rangle^{1/2}$, υ_r is non-random and uniform

in $D\bar{P}$: $\upsilon \partial^2 \bar{U}_j / \partial x_i^2 \Rightarrow (\upsilon + \upsilon_r)\, \partial^2 \bar{U}_j / \partial x_i^2$

hence : LES solution approximates DNS at a smaller Re

effective η scale : $\bar{\eta} \equiv \left[(\upsilon + \upsilon_r)^3 / \varepsilon \right]^{1/4} = \ell_s (1 + \upsilon/\upsilon_r)^2$

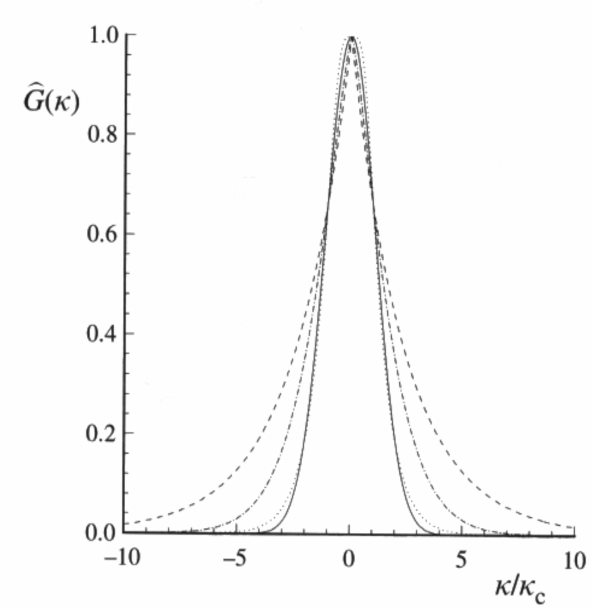
energy spectra : assume to be Kolmogorov as function of $(\eta, \bar{\eta})$

filter transfer : $\hat{G}_s = \exp\left[-3C\kappa^{4/3} (\bar{\eta}^{4/3} - \eta^{4/3}) / 4 \right]$

implies : Pao filter with $\Delta = \ell_s / C_s$

$$C_s \approx 0.15 \left[1 + (7\eta/\Delta)^{4/3} \right]^{1/4}$$

comparisons : (—) Gaussian, (...) Smagorinsky



LES.16 Large Eddy Simulation Model for Turbulent Flow

Filtered NS solutions, limiting cases on filter width Δ

$\Delta\eta << 1$: TS on residual stress tensor shows

$$\tau_{ij}^{R} \equiv \overline{\bar{U}_i \bar{U}_j} - \bar{U}_i \bar{U}_j = \frac{\Delta}{12} \frac{\partial \bar{U}_i}{\partial x_k} \frac{\partial \bar{U}_j}{\partial x_k} + O(\Delta^4)$$

of little practical importance as υ dominates $\upsilon_s \sim C_s^2 (\Delta/\eta)^2$

$\Delta\ell << 1$: turbulence length scale $\ell = k^{3/2}/\varepsilon$,

$$\bar{\boldsymbol{U}} \Rightarrow \langle \boldsymbol{U} \rangle \text{ , } \mathbf{u}' \Rightarrow \mathbf{u}, \tau_{ij}^{R} \Rightarrow \langle u_i u_j \rangle$$

for homogeneous turbulent shear flow, e.g.

$$\upsilon_r \Rightarrow \upsilon^t \text{ , } \ell_s \Rightarrow \ell_{\text{mix}} = \left| \langle u_1 u_2 \rangle \right|^{1/2} \left| \partial \langle U_1 \rangle / \partial x_2 \right|$$

is not a $f(C_s)$ hence $C_s \Rightarrow 0$ as ℓ_{mix}/Δ

laminar flow : τ_{ij}^{R} not necessarily zero in laminar flow

for $C_s = 0$, $\tau_{ij}^{R} \propto \Delta^2$ which is incorrect, hence $C_s = 0$

near-wall issues : LES-NWR is infeasible for high Re applications

LES-NWM for channel flows, no filter in wall normal direction

in wall-normal direction, $\tau_{ij}^{R}(y) \approx \langle u_i u_j(y) \rangle$

$$\therefore \ell_s = C_s \Delta \approx \left[1\text{-exp}\left(y^+ / A^+ \right) \right]$$

LES.17 Large Eddy Simulation Model for Turbulent Flow

CFD Smagorinsky model shortcomings

$$\text{Smagorinsky}: C_s = \begin{cases} 0 \quad , & \text{laminar flow} \\ \sim \omega^2 \,, & \text{near-wall damping} \\ 0.15 \,, & \text{high Re unbounded turbulent flows} \end{cases}$$

A dynamic model utilizes dual filters

$$\text{grid filter}: \ \bar{\Delta} \sim h \qquad , \ \bar{\boldsymbol{U}} \equiv \int (G(|\mathbf{r}|\bar{\Delta})\ \boldsymbol{U}(\mathbf{x}\text{-}\mathbf{r},t)\, \mathrm{d}\mathbf{r}$$

$$\text{test filter}: \ \tilde{\Delta} \equiv 2\bar{\Delta} \qquad , \ \tilde{\bar{\boldsymbol{U}}} \equiv \int (G(|\mathbf{r}|\tilde{\Delta})\ \bar{\boldsymbol{U}}(\mathbf{x}\text{-}\mathbf{r},t)\, \mathrm{d}\mathbf{r}$$

$$\text{test filter transform function}: \ \hat{G}(\kappa;\tilde{\Delta})\ \hat{G}(\kappa;\bar{\Delta}) = \hat{G}(\kappa;(\tilde{\Delta}^2 + \bar{\Delta}^2)^{1/2})$$

$$\text{effect of double filtering}: \ \tilde{\bar{\boldsymbol{U}}} = \int (G(|\mathbf{r}|\ ;\tilde{\Delta})\ \bar{\boldsymbol{U}}(\mathbf{x}\text{-}\mathbf{r},t)\, \mathrm{d}\mathbf{r}$$

$$= \int (G(|\mathbf{r}|\ ;\tilde{\bar{\Delta}})\ \boldsymbol{U}(\mathbf{x}\text{-}\mathbf{r},t)\, \mathrm{d}\mathbf{r}$$

$$\tilde{\bar{\Delta}} = (\tilde{\Delta}^2 + \bar{\Delta}^2)^{1/2} \ \text{for Gaussian}$$

$$\text{resolution}: \ \bar{\boldsymbol{U}} - \tilde{\bar{\boldsymbol{U}}} \propto \text{largest motions } \textit{not} \text{ resolved on grid using } \tilde{\Delta}$$

LES.18 Large Eddy Simulation Model for Turbulent Flow

CFD stabilization acts as an *implicit* filter

$$\overline{D\mathbf{P}}^h: \qquad \frac{\bar{D}\bar{U}_j}{\bar{D}t} = -\frac{1}{\rho_0}\frac{\partial \bar{p}_r}{\partial x_j} + \upsilon \nabla^2 \bar{U}_i - \frac{\partial}{\partial x_i}\left(\tau_{ij}^r + \tau_{ij}^h\right)$$

LES theory: h sufficiently small such that $\tau^h \ll \tau^r$

CFD thinking: assume $\tau^r \approx 0$, let τ^h do the h-scale dissipation

LES-NWM BC: $\bar{U}_2 = 0$ and $\tau_{i2}^r(x,0,z) = f\left(u^+\right)\bar{U}_i\left(x,y_p,z\right)$

Appraisal of LES

is incomplete model, since $\Delta \sim \Delta\left(h,\mathbf{x}\right)$ is an unknown

provides mathematical framework for unsteady turbulent NS analysis

VLES amounts to poorly resolved LES, approaches unsteady RaNS

LES seeks 80% resolution of energy-containing eddies

LES solution can be time-averaged for comparison to steady RaNS

for bounded flows, NWM BCs are a research topic

DES concept addresses this issue