PNS.1 Steady Aerodynamic Flows

Farfield, subsonic-transonic potential flow, $\mathbf{u} = \nabla \cdot \varphi$

D*M*:

$$\mathcal{L}(\phi) = (1 - \mathrm{M}_\infty^2)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

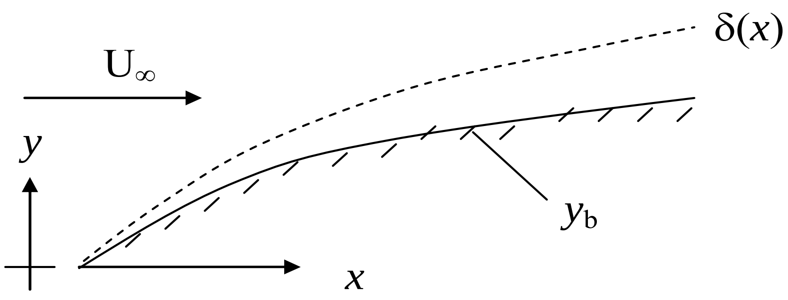
$$\ell(\phi) = \hat{\mathbf{n}} \cdot \nabla \phi - \mathrm{U}_\infty \hat{\mathbf{i}} \cdot \hat{\mathbf{n}} = 0$$

D*E*:

$$p(\mathbf{x}_\delta) = p_\infty - \rho \nabla \phi \cdot \nabla \phi / 2$$

Nearfield, boundary layers wash aerosurfaces

viscous, turbulent effects dominate



D*M*: $\nabla \cdot \mathbf{u} = 0$

D**P**: $\mathbf{u} \cdot \nabla \mathbf{u} = \frac{-1}{\rho_0} \nabla p + \nabla \mathbf{T}$

in the region $y_\mathrm{b}(x) \le y(x) \le \delta(x)$

$\delta(x) \equiv$ boundary layer thickness
$\mathbf{T}$ = stress tensor (viscous + turbulent effects)

PNS.2 Parabolic Navier-Stokes, Boundary Layer Form

Reynolds ordering of steady INS $\Rightarrow n = 2$, subsonic BL

DP_y: pressure through BL is constant
$\Rightarrow P(x)$ from potential farfield DM

DP_x: $\partial^2 u/\partial x^2$ is $O(\delta^2)$, hence negligible, Re $= O(\delta^{-2}) >> 1$
$\Rightarrow$ parabolic PDE on $x \geq x_0$, $0 \leq y \leq \delta(x)$

DM: $\partial v/\partial y = - \partial u/\partial x$, hence initial value on $0 < y \leq \delta(x)$

Laminar - thermal subsonic BL non-D conservation form

$$\mathcal{L}(u) = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \mathrm{Eu}\frac{\mathrm{d}P_I}{\mathrm{d}x} - \frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial y^2} + \frac{\mathrm{Gr}}{\mathrm{Re}^2}\Theta\,\hat{\mathbf{g}}\cdot\hat{\mathbf{i}} = 0$$

$$\mathcal{L}(\Theta) = u\frac{\partial \Theta}{\partial x} + v\frac{\partial \Theta}{\partial y} - \frac{\mathrm{Ec}}{\mathrm{Re}}\left(\frac{\partial u}{\partial y}\right)^2 - \frac{1}{\mathrm{Pe}}\frac{\partial^2 \Theta}{\partial y^2} = 0$$

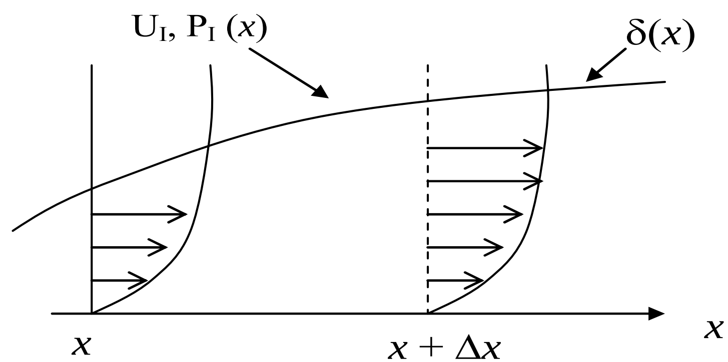
$$\mathcal{L}(v) = \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} = 0$$

BCs:

$$u(x, y=0) = 0 = v(x, y=0)$$

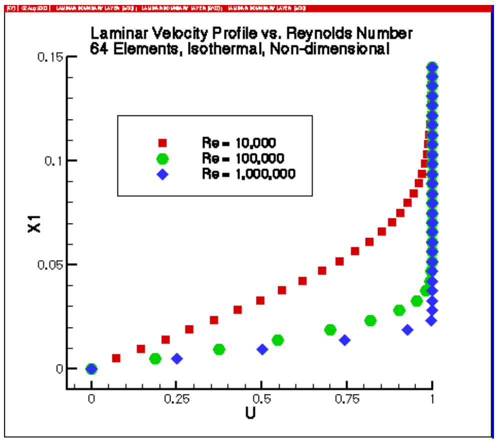
$$\left.\frac{\partial u}{\partial y}\right|_{x, y/\delta>1} = 0, \quad \Theta(x,0) = \Theta_{\mathrm{wall}}$$

$$\left.\frac{\partial \Theta}{\partial y}\right|_{x, y/\delta>1} = 0$$

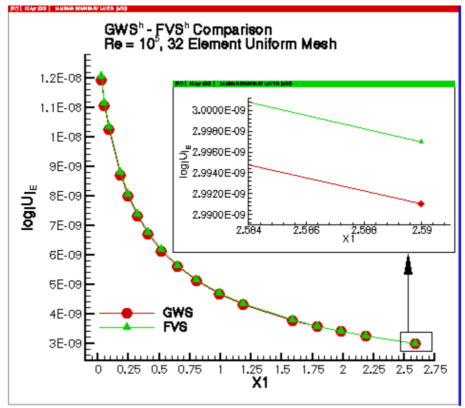


PNS.3 GWS^h, FVS^h + θTS for BL, Solution Nuances

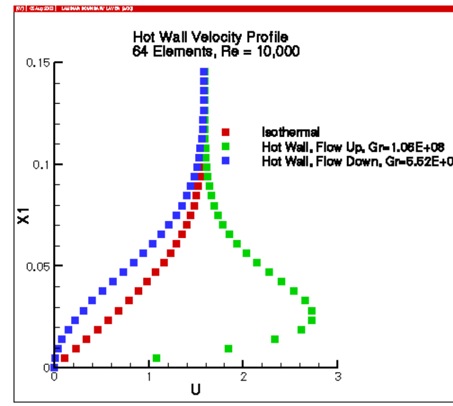
$\{U(n\Delta x)\}$ profiles for Re



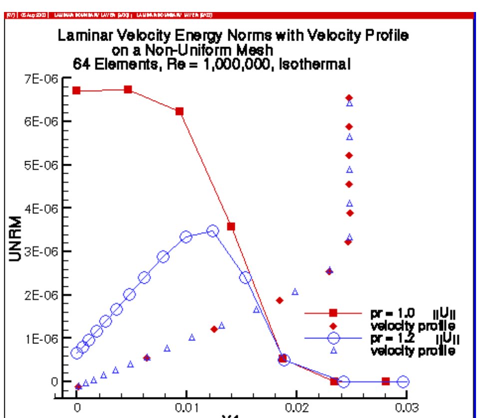
GWS^h optimality



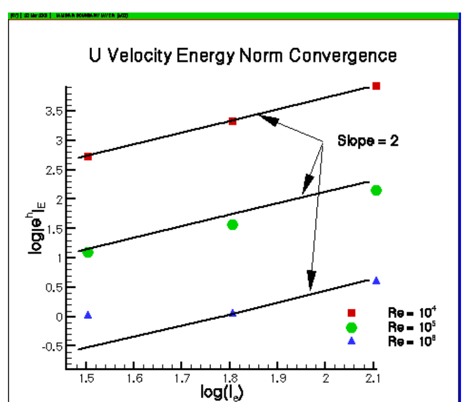
Thermal BLs



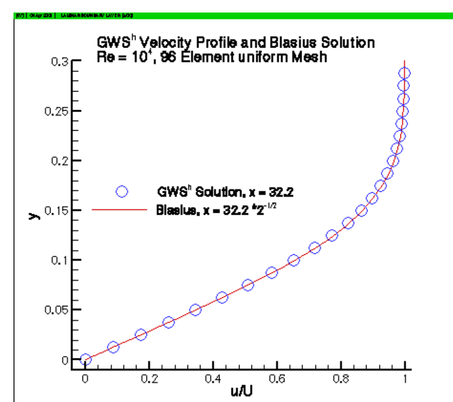
Solution mesh adaptation



GWS^h convergence



GWS^h verification



PNS.4 PNS Boundary Layer Flow, Turbulence

BL form of NS valid only for Re >> 1

aircraft	Mach	U_∞ (m/s)	L (m)	Re	Re/L
commuter	0.3	125	10	3E07	*O*(E06)
wide body	0.9	250	40	2E08	*O*(E06)

BL flows will be turbulent (!)

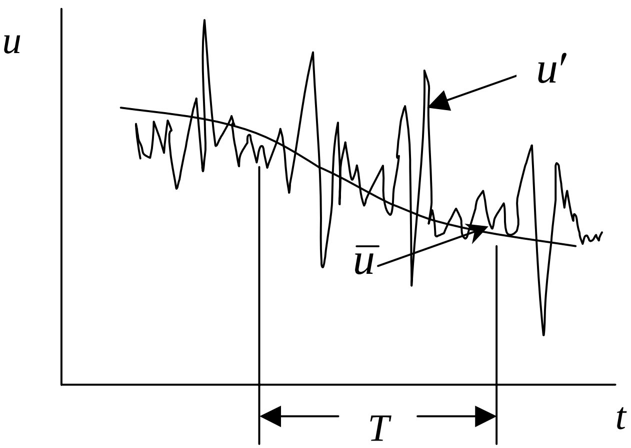
resolution of BL velocity components

$$u(\mathbf{x},t) \equiv \overline{u}(\mathbf{x}) + u'(\mathbf{x},t)$$

time-averaging

$$\overline{u}(\mathbf{x}) \equiv \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} u(\mathbf{x},\tau)\mathrm{d}\tau$$

$$\overline{u'} = 0$$



laboratory data

PNS.5 Turbulent Boundary Layer, Reynolds Stress

Time averaging of BL DM and DP

DM: both terms linear, hence $\nabla \cdot \overline{\mathbf{u}} = 0 = \nabla \cdot \mathbf{u}'$

DP$_x$: non-linear convection term generates a new contribution

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \Rightarrow \frac{\partial}{\partial x}(uu) + \frac{\partial}{\partial y}(vv) \quad \text{via D}M$$

$$\overline{uu} \Rightarrow \overline{u}\ \overline{u} + \overline{u'u'}$$
$$\overline{vu} \Rightarrow \overline{v}\ \overline{u} + \overline{v'u'}$$

Reynolds ordering confirms that $O\left(\overline{u'u'}\right) \approx O\left(\overline{v'u'}\right) \approx O(\delta)$

$$\frac{\partial}{\partial x}\left(\overline{u}\ \overline{u} + \overline{u'u'}\right) \Rightarrow O\left(1 \cdot 1/1 + \delta/1\right)$$
$$\frac{\partial}{\partial y}\left(\overline{v}\ \overline{u} + \overline{v'u'}\right) \Rightarrow O\left(\delta \cdot 1/\delta + \delta/\delta\right)$$

hence:

Reynolds normal stress $\overline{u'u'}$ contribution negligible

Reynolds shear stress $\overline{v'u'}$ contribution must be included

PNS.6 Boundary Layer Flow, Turbulence Modeling

Reynolds kinematic shear stress modeled after Stokes

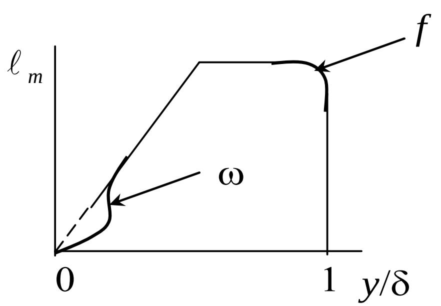
$$\overline{v'u'} \equiv -\upsilon^t \frac{\partial \bar{u}}{\partial y} \;,\; \upsilon^t \equiv \text{turbulent "eddy" viscosity, units} \, (\mu/\rho_\infty = \upsilon) \Rightarrow (\mathrm{L}^2/t)$$

Prandtl mixing length model

$$\upsilon^t \equiv (\omega \ell_m)^2 \left| \frac{\partial \bar{u}}{\partial y} \right| f \Rightarrow (\mathrm{L}^2)(1/t)$$

where: $\ell_m \equiv$ mixing length

$\omega, f =$ near wall, freestream damping



Turbulent kinetic energy-dissipation model

$$\upsilon^t \equiv \mathrm{C}_\mu k^2/\varepsilon \Rightarrow (\mathrm{L}/t)^4 (t^3/\mathrm{L}^2)$$

where:

$$k \equiv \frac{1}{2}\left(\overline{\mathbf{u}' \cdot \mathbf{u}'}\right) = \frac{1}{2}\left(\overline{u'u'} + \overline{v'v'} + \overline{w'w'}\right)$$

$$\varepsilon \equiv \frac{2\upsilon}{3}\overline{\left(\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_i}{\partial x_j}\right)}\delta_{jk}$$

and: $\mathcal{L}(k)$ and $\mathcal{L}(\varepsilon)$ BL forms augment BL DM & DP$_x$

PNS.7 GWS^h + θTS, Turbulent BL, MLT Closure

Turbulent BL conservation law form, time-averaged $q(x,y)$, MLT

$$\mathrm{D}\mathbf{P}_x : \mathcal{L}(\bar{u}) = \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} - \frac{1}{\mathrm{Re}}\frac{\partial}{\partial y}(1+\mathrm{Re}^t)\frac{\partial \bar{u}}{\partial y} + \frac{\mathrm{dP}^I}{\mathrm{d}x} + \frac{\mathrm{Gr}}{\mathrm{Re}^2}\bar{\Theta}\hat{\mathbf{g}}\cdot\hat{\mathbf{i}} = 0$$

$$\mathrm{D}\Theta : \mathcal{L}(\bar{\Theta}) = \bar{u}\frac{\partial \bar{\Theta}}{\partial x} + \bar{v}\frac{\partial \bar{\Theta}}{\partial y} - \frac{1}{\mathrm{Re}}\frac{\partial}{\partial y}\left(\frac{1}{\mathrm{Pr}} + \frac{\mathrm{Re}^t}{\mathrm{Pr}^t}\right)\frac{\partial \bar{\Theta}}{\partial y} - \frac{\mathrm{Ec}}{\mathrm{Re}}\left(\frac{\partial \bar{u}}{\partial y}\right)^2 \equiv 0$$

$$\mathrm{D}M : \mathcal{L}(\bar{v}) = \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{u}}{\partial x} = 0$$

$$\mathrm{D}\mathbf{P}_y : \mathcal{L}(\bar{p}) = \frac{\partial}{\partial y}\left(\rho_0 P^I + \overline{v'}\,\overline{v'}\right) = 0$$

where: $\mathrm{Re}^t \equiv (\upsilon^t/\upsilon)_{\mathrm{dim}}$ = turbulent Reynolds number

$$\upsilon^t \equiv (\omega\ell_m)^2\left|\frac{\partial \bar{u}}{\partial y}\right| f = \text{MLT eddy viscosity}$$

$$\omega = 1 - \exp(-y/\mathrm{A}) = \text{van Driest damping},\ \ \mathrm{A} \approx 25$$

$$\ell_m = \text{Prandtl mixing length} = \begin{cases} \kappa y, & on\ \ 0 \le y/\delta \le \lambda/\kappa \\ \lambda\delta, & on\ \ \lambda/\kappa < y/\delta \le 1 \end{cases} \quad \begin{matrix} \kappa = 0.405 \\ \lambda = 0.09 \end{matrix}$$

$$f = \left[1 + 5.5(y/\delta)^6\right]^{-1} = \text{Klebanoff damping}$$

$\mathrm{Pr}^t \cong \mathrm{Pr}$ for turbulent Prandtl number (usually)

$\overline{v'}\,\overline{v'}$ = Reynolds transverse normal stress

PNS.8 GWS^h + θTS Performance, Turbulent BL, MLT Closure

Accuracy, convergence, *regular* non-uniform Ω^h refinement

theory:

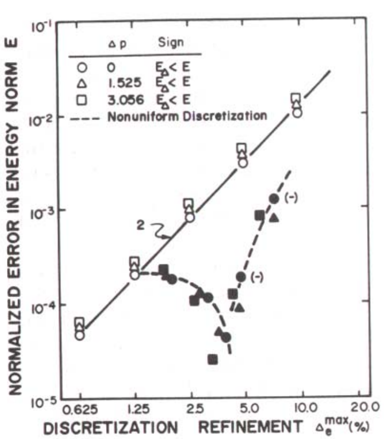
$$\left| e^h(n\Delta x)\right|_E \le C h_e^{2\gamma}\left\|\mathrm{data}\right\|^2_{H^{k-1}} + C_x \Delta x^3 \left\| U_0 \right\|^2_{H1},\ \gamma = \min\,(k, r-1)$$

norm:

$$\left| u^h(n\Delta x)\right|_E \equiv \frac{1}{2}\int_\Omega \upsilon^t \left(\frac{\partial u^h}{\partial y}\right)^2 dy = \frac{1}{2}\sum_{\Omega^h}\int_{\Omega_e}(\cdot)dy$$

$$= \frac{1}{2\,\mathrm{Re}}\sum_e^M \{U\}_e^T\{\mathrm{RET}\}_e^T[\mathrm{A3011}]\{U\}_e$$

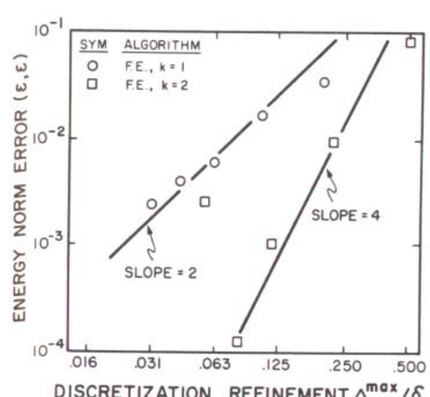
IC, M = 80 laminar



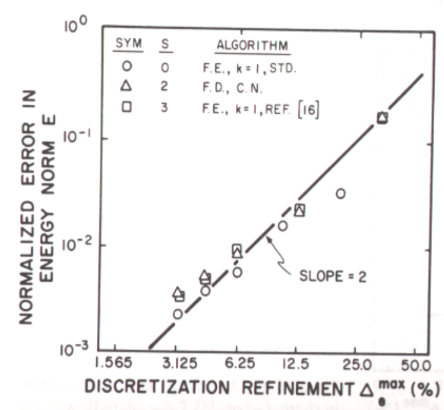
Ω^h progressions

Linear basis, $k=1$		Quadratic basis, $k=2$	
M	ρ	M	ρ
12	1.627	6	2.814
24	1.222	12	1.510
36	1.125	18	1.271
48	1.083	24	1.176
60	1.061	30	1.110

Convergence



Optimality



PNS.9 Turbulent Boundary Layer, TKE Closure

Turbulent kinetic energy-isotropic dissipation closure model

eddy viscosity: $\upsilon^t \equiv C_\mu k^2 / \varepsilon\,,\ \ C_\mu = 0.09$

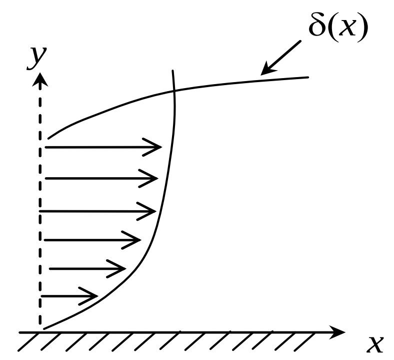
$$k \equiv \frac{1}{2}\left(\overline{\mathbf{u}' \cdot \mathbf{u}'}\right) = \frac{1}{2}\left(\overline{u'u'} + \overline{v'v'} + \overline{w'w'}\right) \quad \varepsilon \equiv \frac{2\nu}{3}\left(\overline{\frac{\partial u_i'}{\partial x_k}}\,\overline{\frac{\partial u_i'}{\partial x_j}}\right)\delta_{jk}$$

$\mathcal{L}(k, \varepsilon)$ conservation PDEs, non-D BL form

$$\mathcal{L}(k) = u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} - \frac{1}{\mathrm{Pe}}\frac{\partial}{\partial y}\left(1 + \frac{\mathrm{Re}^t}{\mathrm{C}_k}\right)\frac{\partial k}{\partial y} - \tau_{12}\frac{\partial u}{\partial y} + \varepsilon = 0$$

$$\mathcal{L}(\varepsilon) = u\frac{\partial \varepsilon}{\partial x} + v\frac{\partial \varepsilon}{\partial y} - \frac{1}{\mathrm{Pe}}\frac{\partial}{\partial y}\left(1 + \frac{\mathrm{Re}^t}{\mathrm{C}_\varepsilon}\right)\frac{\partial \varepsilon}{\partial y} - \mathrm{C}_\varepsilon^1\frac{\varepsilon}{k}\tau_{12}\frac{\partial u}{\partial y} + \mathrm{C}_\varepsilon^2\frac{\varepsilon}{k}\varepsilon = 0$$

$$\mathcal{L}(\tau_{12}) = \tau_{12} + \upsilon^t\frac{\partial u}{\partial y} = \tau_{12} + \mathrm{C}_\mu\frac{k^2}{\varepsilon}\frac{\partial u}{\partial y} = 0$$



TKE model adds non-linear parabolic PDE + BCs + IC pair

BCs: $k(x, y=0) = 0,\ \ \varepsilon(x, y=0) \Rightarrow \varepsilon_w < \infty$

$$\left.\frac{\partial k}{\partial y}, \frac{\partial \varepsilon}{\partial y}\right|_{y \geq \delta(x)} = 0$$

IC: $k(x_0, y) = ? = \varepsilon(x_0, y)$

PNS.10 TKE for Turbulent BL, Near-Wall Corrections

TKE closure model requires near-wall corrections

low Re^t closure model constant modifications (Lam-Bremhorst)

$$\upsilon^t \Rightarrow f_v C_\mu k^2/\varepsilon: \; f_v = \left(1-\exp(-0.0165 R_y)\right)^2 \left(1+20.5/\mathrm{Re}^t\right)$$
$$C_\varepsilon^1 \Rightarrow f_k C_\varepsilon^1 \quad : \; f_k = (1+0.05 f_v^{-1})^3$$
$$C_\varepsilon^2 = f_\varepsilon C_\varepsilon^2 \quad : \; f_\varepsilon = 1-\exp(-\mathrm{Re}^t)^2$$

$$\mathrm{Re}^t = \upsilon^t/\upsilon$$
$$R_y = k^{1/2} y/\upsilon$$

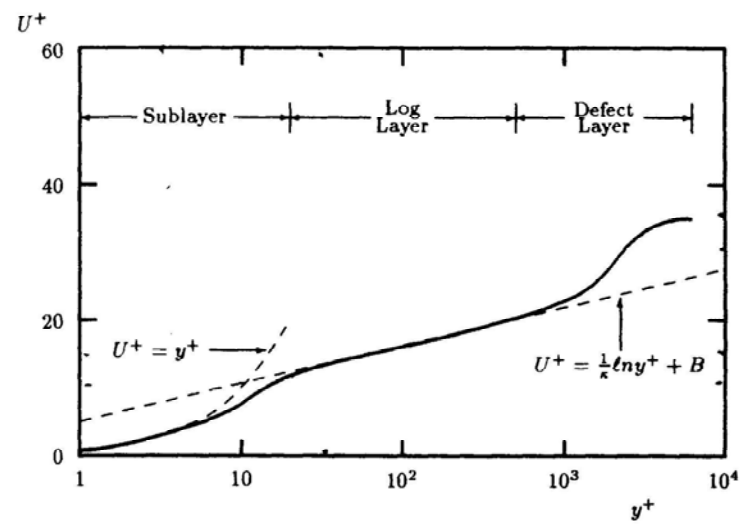
BL similarity TKE variable distributions as *f* (*y*)

$$U^+ \equiv U/u_\tau = \kappa^{-1}\log(y^+ E) + B$$
$$y^+ \equiv u_\tau y/\upsilon$$

near-wall production = dissipation in $\mathcal{L}(k)$

$\Rightarrow$

$$\upsilon^t = \kappa y u_\tau$$
$$k = u_\tau^2 / C_\mu$$
$$\varepsilon = (\kappa y)^{-1}|u_\tau|^3$$
$$\tau_w = \sqrt{C_\mu}k = u_\tau (C_\mu)^{-1/2}$$



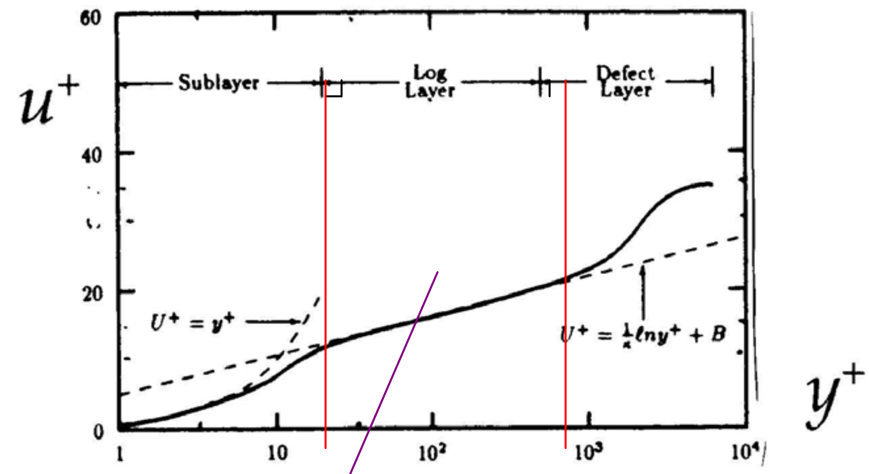
PNS.10A Turbulent Boundary Layer Similarity

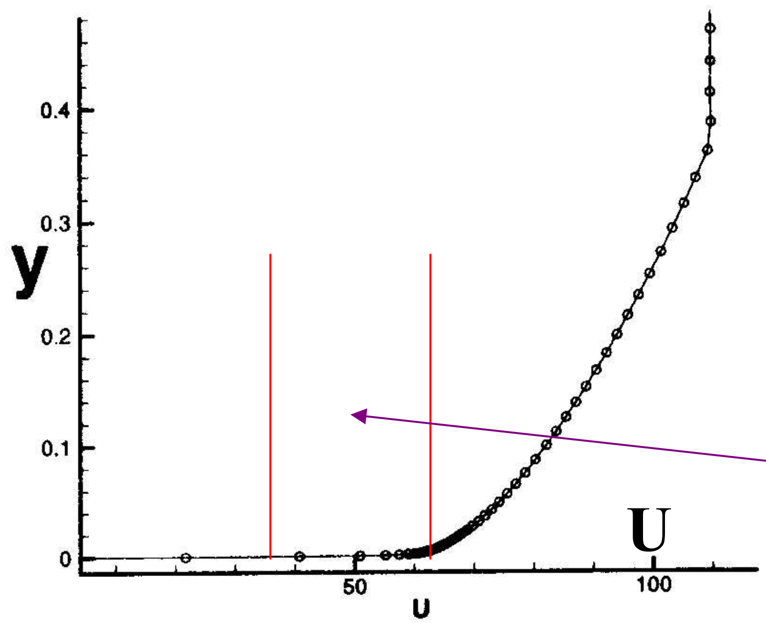
Turbulent BL similarity

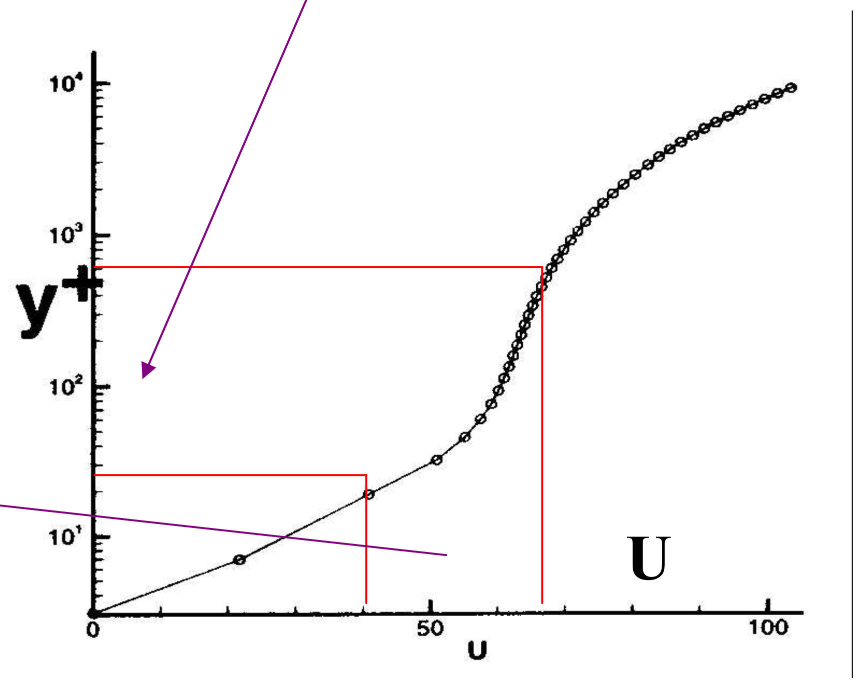
$$U/u_\tau = \frac{1}{\kappa}\,\ell n\left(y^+ E\right) + B \equiv U^+$$

$$y^+ = u_\tau y/\upsilon$$

example: IDENT 2400 boundary layer







PNS.11 $\mathrm{GWS}^h + \theta\mathrm{TS}$ Template, BL, TKE + Low Re^t

Template pseudo-code modifications ($\{FV\}_e$ unchanged)

for $\{Q\}_e = \begin{Bmatrix} U \\ T \\ K \\ E \end{Bmatrix}_e$:

$$\begin{aligned}\{\mathrm{FQ}\}_e &= (\ )(\ )\{\overline{U}\}(1)[\mathrm{A3000}]\{Q\mathrm{P} - \mathrm{QN}\} \\ &\quad + (\Delta x/2)(\ )\{V\mathrm{P}, \mathrm{VN}\}(0)[\mathrm{A3001}]\{Q\mathrm{P}, \mathrm{QN}\} \\ &\quad + (\Delta x/2, \mathrm{Pa}^{-1})(\ )\{\mathrm{RET}\}[\mathrm{A3011}]\{Q\mathrm{P}, \mathrm{QN}\} + \{\mathrm{b}(Q)\}\end{aligned}$$

Source terms $\{\mathrm{b}\,(\cdot)\}$ unchanged for $\{Q\} = \{U, T\}^T$, and

$$\begin{aligned}\{\mathrm{b}(K)\}_e &= \int_{\Omega_e} \{N\}(\tau_{12}\partial u_e/\partial y)\mathrm{d}y + \int_{\Omega_e} \{N\}\varepsilon_e \mathrm{d}y \\ &= (\Delta x/2)(\ )\{TXY\}(0)[\mathrm{A3001}]\{U\} + (\Delta x/2)(\ )\{\ \}(1)[\mathrm{A200}]\{E\} \\ \{\mathrm{b}(E)\}_e &= \mathrm{C}_\varepsilon^1 \int_{\Omega_e} \{N\}(\tau_{12}(\varepsilon/k)_e \partial u_e/\partial y)\mathrm{d}y + \mathrm{C}_\varepsilon^2 \int_{\Omega_e} \{N\}(\varepsilon/k)_e \varepsilon_e \mathrm{d}y \\ &= (\Delta x/2, \mathrm{CE1})(\mathrm{FE1})\{TXY, E/K\}(0)[\mathrm{A3001}]\{E\} \\ &\quad + (\Delta x/2, \mathrm{CE2})(\mathrm{FE2})\{E/K\}(1)[\mathrm{A3000}]\{E\}\end{aligned}$$

Reynolds algebraic shear stress model template

$$\begin{aligned}\{\mathrm{F}TXY\}_e &= (\ )(\ )\{\ \}(1)[\mathrm{A200}]\{TXY\} \\ &\quad + (\mathrm{Re}^{-1})(\mathrm{FNU})\{\mathrm{RET}\}(0)[\mathrm{A3001}]\{U\}\end{aligned}$$

PNS.12 GWS^h + θTS TKE BL, Quasi-Newton Jacobian

Size, deeply embedded non-linearity precludes Newton

quasi-Newton jacobians:

$$\begin{bmatrix} \mathrm{JUU}, & \mathrm{JUV}, & \mathrm{JUT} \\ \mathrm{JVU}, & \mathrm{JVV}, & 0 \\ \mathrm{JTU}, & \mathrm{JTV}, & \mathrm{JTT} \end{bmatrix}_e , \begin{bmatrix} \mathrm{JKK}, & \mathrm{JKE}, & \mathrm{JKT}_{xy} \\ \mathrm{JEK}, & \mathrm{JEE}, & \mathrm{JET}_{xy} \\ \mathrm{JT}_{xy}\mathrm{K}, & \mathrm{JT}_{xy}\mathrm{E}, & \mathrm{JT}_{xy}\mathrm{T}_{xy} \end{bmatrix}_e$$

solution sequence:

$\{\delta U, \delta V, \delta T\}^{p+1}$ unchanged from laminar, MLT
update $\{U, V, T\}^{p+1}$
$\{\delta K, \delta E, \delta T_{xy}\}^{p+1}$ uses $\{U, V, T\}^{p+1}$
update $\{K, E, T\}^{p+1}$
index p, return to $\{\delta U, \delta V, \delta T\}^{p+1}$

oscillating convergence:

use {RETN} in $\{\mathrm{F}U, \mathrm{F}T\}^p$
use {UN, VN} in $\{\mathrm{F}K, \mathrm{F}E, \mathrm{F}T_{xy}\}^p$

templates:

$[\mathrm{JAC}]_e$ for $\{\mathrm{F}U, \mathrm{F}V, \mathrm{F}T\}$ are unchanged
$[\mathrm{JAC}]_e$ for $\{\mathrm{F}K, \mathrm{F}E, \mathrm{F}T_{xy}\}$ fully utilizes chain rule

PNS.12A GWS^h + θTS TKE Closure Jacobian Coupling

Jacobian coupling for convection terms is unchanged

$$\text{diffusion term}: \mathcal{L}(q) \Rightarrow -\frac{1}{\mathrm{Re}}\frac{\partial}{\partial y}\left(\frac{1}{\mathrm{Pr}}+\frac{\mathrm{Re}^t}{\mathrm{C}_q\,\mathrm{Pr}^t}\right)\frac{\partial q}{\partial y}\ ,\ q=\{k,\ \varepsilon\}$$

$$\frac{\partial}{\partial q}(\cdot)=\frac{1}{\mathrm{Re}}\frac{\partial}{\partial y}\left(1+\frac{\mathrm{Re}^t}{\mathrm{C}_q}\right)\frac{\partial(\cdot)}{\partial y}+\frac{1}{\mathrm{Re}}\frac{\partial}{\partial y}\left(\frac{\partial \tau_{12}^e}{\partial q}\right)\frac{\partial q}{\partial y}$$

$$\frac{\partial \tau_{12}}{\partial q}=\frac{\partial}{\partial q}\left(\mathrm{C}_\mu f_\upsilon k^2/\varepsilon\right) \Rightarrow \begin{cases} 2\mathrm{C}_\mu f_\upsilon\, k/\varepsilon = 2\tau_{12}k^{-1} \\ -\mathrm{C}_\mu f_\upsilon\,(k/\varepsilon)^2 = -\tau_{12}\varepsilon^{-1} \end{cases}$$

assuming $\mathrm{Pr}^t \approx \mathrm{Pr}$, hence $\mathrm{RePr} \approx \mathrm{RePr}^t = \mathrm{Pe}$

$$\begin{aligned}[\mathrm{J}KK]_e = &(\Delta x/2, \mathrm{Pe}^{-1})(\ )\{\ \}(-1)[\mathrm{A211}][\ \ ] \\ &+(\Delta x/2, \mathrm{Pe}^{-1}, \mathrm{C}_k^{-1})(\ )\{\mathrm{RET}\}(-1)[\mathrm{A3011}][\ \ ] \\ &+(\Delta x, \mathrm{Pe}^{-1}, \mathrm{C}_k^{-1})(\ )\{K\}(-1)[\mathrm{A3110}][\mathrm{RET}, K^{-1}] \\ &+(\Delta x)(\ )\{U\}(0)[\mathrm{A3100}][T\mathrm{XY}, K^{-1}]\end{aligned}$$

PNS.12B GWS^h + θTS TKE Closure Jacobian Coupling

Continuing with jacobians

$$
\begin{aligned}
[\mathrm{J}KE]_e = &(-\Delta x/2, \mathrm{Pe}^{-1}, \mathrm{C}_k^{-1})(\ \ )\{K\}(-1)[\mathrm{A}3110][\mathrm{RET}, E^{-1}] \\
&+(-\Delta x/2)(\ \ )\{U\}(0)[\mathrm{A}3100][T\mathrm{XY}, E^{-1}] \\
&+(\Delta x/2)(\ \ )\{\ \ \}(1)[\mathrm{A}200][\ \ ]
\end{aligned}
$$

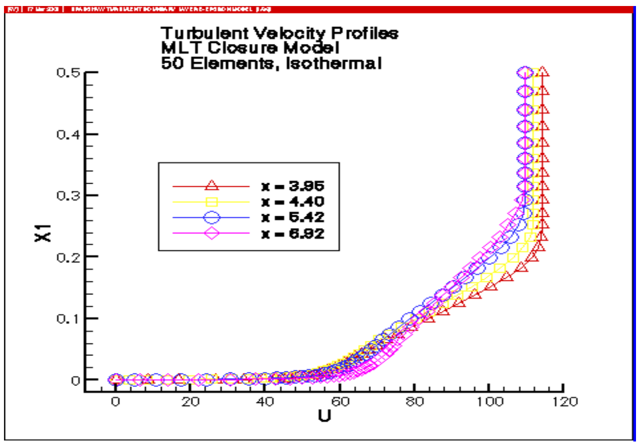
$$
\begin{aligned}
[\mathrm{J}EK]_e = &(\Delta x, \mathrm{Pe}^{-1}, \mathrm{C}_\varepsilon^{-1})(\ \ )\{E\}(-1)[\mathrm{A}3110][\mathrm{RET}, K^{-1}] \\
&+(-\Delta x/2, \mathrm{C}_\varepsilon^1)(\mathrm{FE}1)\{U\}(0)[\mathrm{A}3100][T\mathrm{XY}, E/K^2] \\
&+(\Delta x, \mathrm{C}_\varepsilon^1)(\mathrm{FE}1)\{U\}(0)[\mathrm{A}3100][T\mathrm{XY}, (E/K)^2] \\
&+(-\Delta x/2, \mathrm{C}_\varepsilon^2)(\mathrm{FE}2)\{\mathrm{E}\}(1)[\mathrm{A}3000][E/K^2]
\end{aligned}
$$

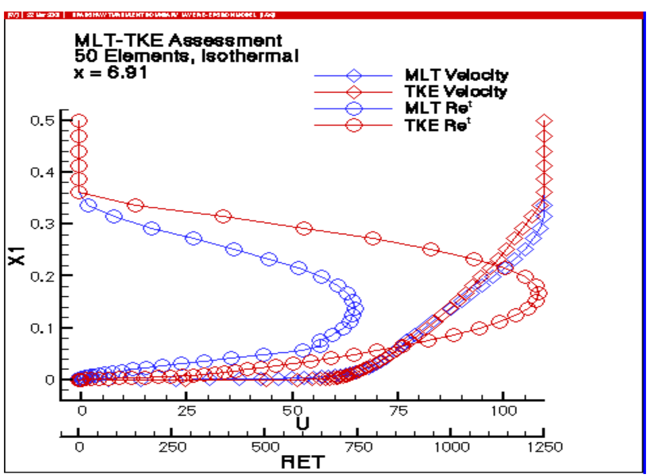
$$
\begin{aligned}
[\mathrm{J}EE]_e = &(\Delta x/2, \mathrm{Pe}^{-1})(\ \ )\{\ \ \}(-1)[\mathrm{A}211][\ \ ] \\
&+(\Delta x/2, \mathrm{Pe}^{-1}, \mathrm{C}_\varepsilon^1)(\ \ )\{\mathrm{RET}\}(-1)[\mathrm{A}3011][\ \ ] \\
&+(\Delta x/2, \mathrm{C}_\varepsilon^2)(\mathrm{FE}2)\{E/K\}(1)[\mathrm{A}3000][\ \ ] \\
&+(\Delta x/2, \mathrm{C}_\varepsilon^2)(\mathrm{FE}2)\{E\}(1)[\mathrm{A}3000][K^{-1}]
\end{aligned}
$$

PNS.13 GWS^h + θTS TKE BL, Validation

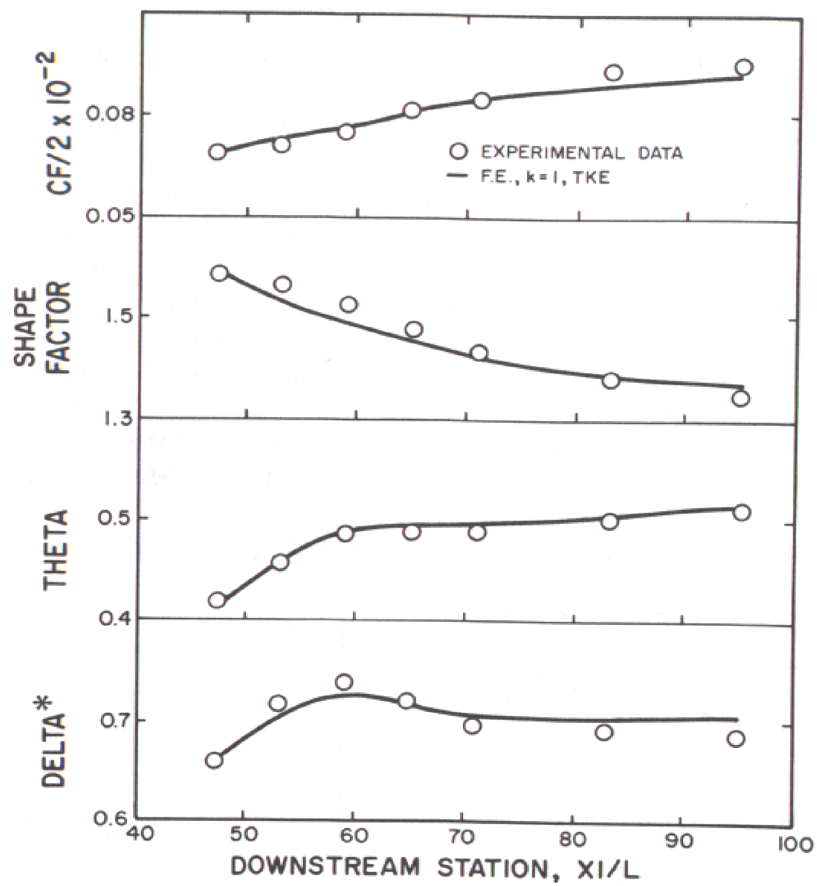
Validation, Bradshaw I 2400 experiment, Re/L≈10^5

MLT, TKE closure solutions





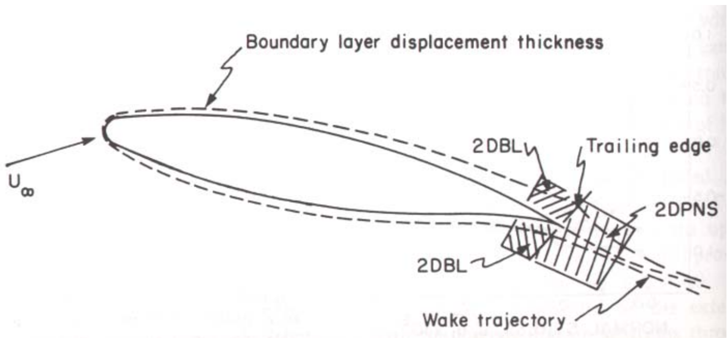
BL integral norm evolutions



PNS.14 Merging Turbulent Boundary Layers

Reynolds-ordered PNS PDE+BCs for merging BLs

Problem statement geometry



BL $\Rightarrow$ PNS theory modifications

DM BCs not valid for ODE on $\{V(y)\}$

$\Rightarrow \nabla \cdot \mathbf{u} = 0$ is now a differential constraint

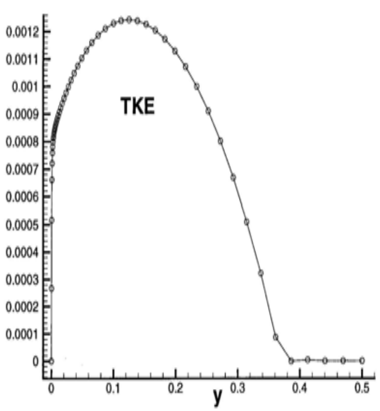
D$\mathbf{P}_x$ remains as developed

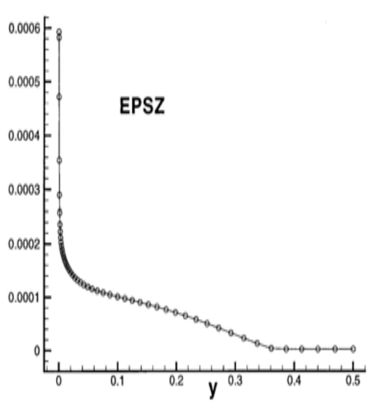
D$\mathbf{P}_y$ still $O(\delta)$, but must be included for BCs

DK, DE remain as developed

$\nabla \cdot$D$\mathbf{P}$ yields pressure Poisson equation

$\Rightarrow$ complementary + particular solutions





BL distributions merging at TE

$\max(\partial k / \partial y, \partial \varepsilon / \partial y)\big|_{\partial\Omega} \Rightarrow$ interior to Ω!

requires Reynolds stress algebraic model

PNS.15 Merging Turbulent Boundary Layers, Reynolds Stress Model

Algebraic Reynolds stress model for uni-directional flows

$$\overline{u_i' u_j'} = k\alpha_{ij} - C_4 k^2/\varepsilon S_{ij} - C_2 C_4 k^3/\varepsilon^2 S_{ik} S_{kj}\ ,\quad S_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$$

for $n = 2$ BL:

$$\overline{u'}\,\overline{u'} = \overbrace{C_1 k - C_2 C_4 \frac{k^3}{\varepsilon^2}\left(\frac{\partial \overline{u}}{\partial y}\right)^2}^{O(\delta)} - \overbrace{2C_4 \frac{k^2}{\varepsilon}\left(\frac{\partial \overline{u}}{\partial x}\right)}^{O(\delta^2)}$$

$$\overline{v'}\,\overline{v'} = C_3 k - C_2 C_4 \frac{k^3}{\varepsilon^2}\left(\frac{\partial \overline{u}}{\partial y}\right)^2 - 2C_4 \frac{k^2}{\varepsilon}\left(\frac{\partial \overline{v}}{\partial y}\right)$$

$$\overline{w'}\,\overline{w'} = C_3 k$$

$$\overline{u'}\,\overline{v'} = C_2 \frac{k^2}{\varepsilon}\left(\frac{\partial \overline{u}}{\partial y}\right)$$

closure model constants:

$C_{01} \approx 2.8,\ C_{02} \approx 0.45$

$$C_1 \equiv \frac{22(C_{01} - 1) - 6(4C_{02} - 5)}{33(C_{01} - 2C_{02})}$$

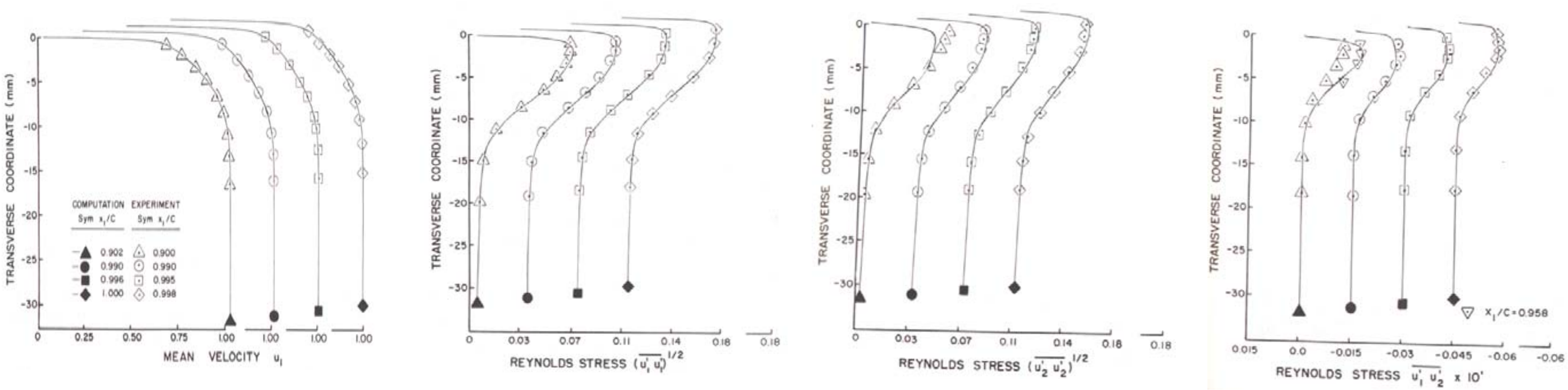
$$C_2 \equiv \frac{4(3C_{02} - 1)}{11(C_{01} - 2C_{02})}$$

$$C_3 \equiv \frac{22(C_{01} - 1) - 12(3C_{02} - 1)}{33(C_{01} - 2C_{02})}$$

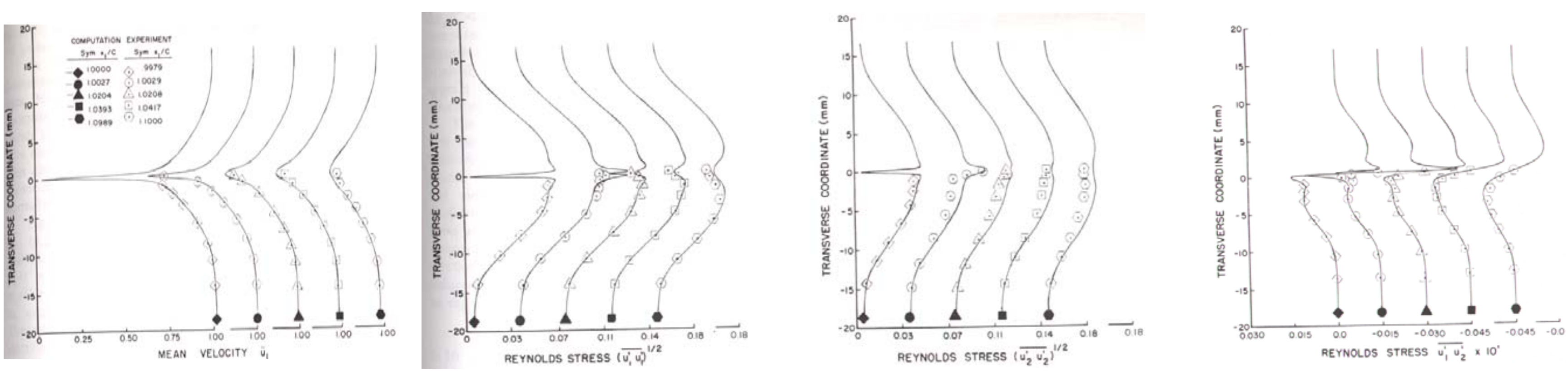
$$C_4 \equiv \frac{44C_{02} - 22C_{01}C_{02} - 128C_{02} - 36C_{02}^2 + 10}{165(C_{01} - 2C_{02})^2}$$

PNS.16 Validation, Merging Turbulent Boundary Layers

GWS^h+θTS BL comparison to data, $0.90 \leq x/c \leq 0.998$



GWS^h+θTS PNS wake comparison to data, $1.00 \leq x/c \leq 1.099$

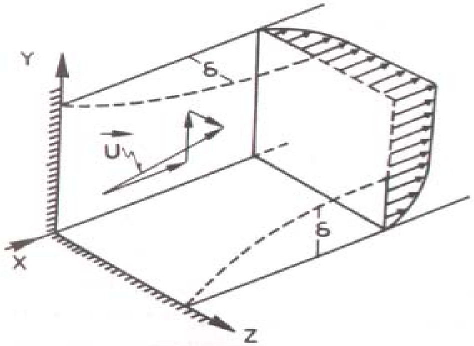


PNS.17 Unidirectional 3-D Turbulent Flows

Algebraic Reynolds stress model for uni-directional flows

$$\overline{u_i' u_j'} = k\alpha_{ij} - C_4 k^2/\varepsilon S_{ij} - C_2 C_4 k^3/\varepsilon^2 S_{ik} S_{kj}\ , \quad S_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$$

flow geometry



for n = 2 BL:

$$\overline{u'u'} = \overbrace{C_1 k - C_2 C_4 \frac{k^3}{\varepsilon^2}\left(\frac{\partial \bar u}{\partial y}\right)^2}^{O(\delta)} \overbrace{- 2C_4 \frac{k^2}{\varepsilon}\left(\frac{\partial \bar u}{\partial x}\right)}^{O(\delta^2)}$$

$$\overline{v'v'} = C_3 k - C_2 C_4 \frac{k^3}{\varepsilon^2}\left(\frac{\partial \bar u}{\partial y}\right)^2 - 2C_4 \frac{k^2}{\varepsilon}\left(\frac{\partial \bar v}{\partial y}\right)$$

$$\overline{w'w'} = C_3 k$$

$$\overline{u'v'} = C_2 \frac{k^2}{\varepsilon}\left(\frac{\partial \bar u}{\partial y}\right)$$

for n = 3 PNS:

$$\overline{u_1' u_1'} = \overbrace{C_1 k - C_2 C_4 \frac{k^3}{\varepsilon^2}\left[\left(\frac{\partial \tilde u_1}{\partial x_2}\right)^2 + \left(\frac{\partial \tilde u_1}{\partial x_3}\right)^2\right]}^{O(\delta)} \overbrace{- 2C_4 \frac{k^2}{\varepsilon}\left[\frac{\partial \tilde u_1}{\partial x_1}\right]}^{O(\delta^2)}$$

$$\overline{u_2' u_2'} = C_3 k - C_2 C_4 \frac{k^3}{\varepsilon^2}\left[\frac{\partial \tilde u_1}{\partial x_2}\right]^2 - 2C_4 \frac{k^2}{\varepsilon}\left[\frac{\partial \tilde u_2}{\partial x_2}\right]$$

$$\overline{u_3' u_3'} = C_3 k - C_2 C_4 \frac{k^3}{\varepsilon^2}\left[\frac{\partial \tilde u_1}{\partial x_3}\right]^2 - 2C_4 \frac{k^2}{\varepsilon}\left[\frac{\partial \tilde u_3}{\partial x_3}\right]$$

$$\overline{u_1' u_2'} = -C_4 \frac{k^2}{\varepsilon}\left[\frac{\partial \tilde u_1}{\partial x_2}\right] - C_2 C_4 \frac{k^3}{\varepsilon^2}\left[\frac{\partial \tilde u_1}{\partial x_3}\left(\frac{\partial \tilde u_2}{\partial x_3}\right)^2 + \left(\frac{\partial \tilde u_3}{\partial x_2}\right)^2 + 2\frac{\partial \tilde u_1}{\partial x_2}\left(\frac{\partial \tilde u_1}{\partial x_1} + \frac{\partial \tilde u_2}{\partial x_2}\right)\right]$$

$$\overline{u_1' u_3'} = -C_4 \frac{k^2}{\varepsilon}\left[\frac{\partial \tilde u_1}{\partial x_3}\right] - C_2 C_4 \frac{k^3}{\varepsilon^2}\left[\frac{\partial \tilde u_1}{\partial x_2}\left(\frac{\partial \tilde u_2}{\partial x_3}\right)^2 + \left(\frac{\partial \tilde u_3}{\partial x_2}\right)^2 + 2\frac{\partial \tilde u_1}{\partial x_3}\left(\frac{\partial \tilde u_1}{\partial x_1} + \frac{\partial \tilde u_3}{\partial x_3}\right)\right]$$

$$\overline{u_2' u_3'} = -C_2 C_4 \frac{k^3}{\varepsilon^2}\left[\frac{\partial \tilde u_1}{\partial x_2}\frac{\partial \tilde u_1}{\partial x_3}\right] - C_4 \frac{k^2}{\varepsilon}\left(\frac{\partial \tilde u_2}{\partial x_3} + \frac{\partial \tilde u_3}{\partial x_2}\right)$$

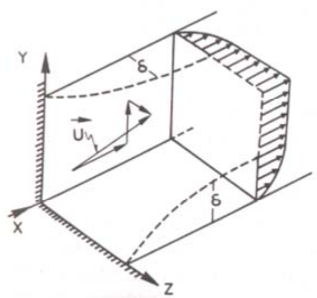
PNS.18 3D Turbulent Duct Flow, Validation

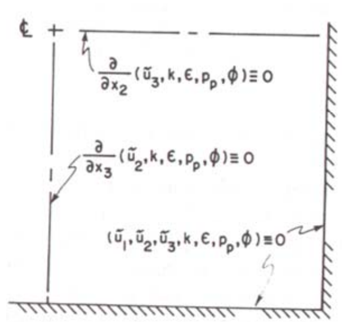
3D PNS algorithm based on PPNS formulation

$$\mathrm{D}M : \nabla^h \cdot \bar{\rho}\tilde{\mathbf{u}}^h \approx 0 \Rightarrow \mathcal{L}(\phi) = -\nabla^2\phi - \nabla\cdot\bar{\rho}\tilde{\mathbf{u}} = 0 + \mathrm{BCs}$$

$$\nabla\cdot\mathrm{D}\mathbf{P} : \nabla\cdot\mathcal{L}(\tilde{\mathbf{u}}) = 0 \Rightarrow \mathcal{L}(p) = -\nabla\cdot\bar{\rho}\ \ \nabla p + s(\bar{\rho},\tilde{\mathbf{u}}) = 0 + \mathrm{BCs}$$

3D square duct flow, BCs



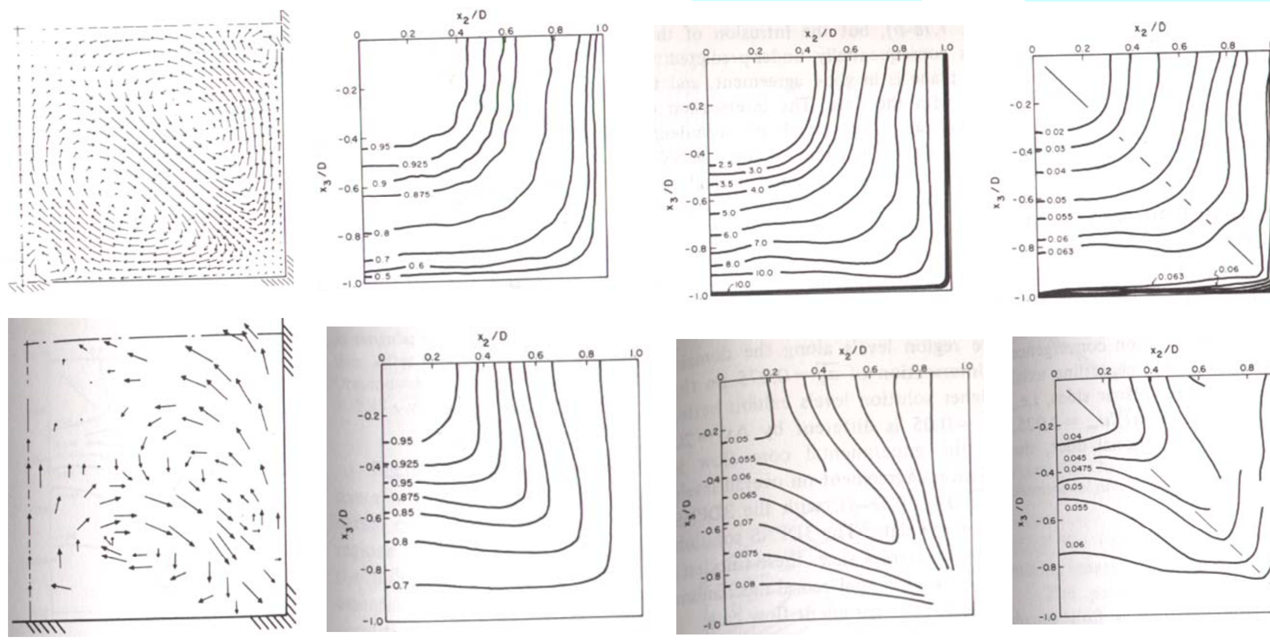


3D PNS GWS^h + θTS solution validation

transverse $\bar{u}_k$ | axial $\bar{u}_1$ | stress $\overline{u_1' u_1'}$ | stress $\overline{u_2' u_2'}$

GWS^h + θTS

experiment



PNS.19 Summary, Steady Turbulent Parabolic Navier-Stokes

Aerodynamic flows ⇔ weak interaction

streamline shapes
- flowfield is uni-directional
- pressure impressed from farfield

large Reynolds number, $Re/L > 10^6$
- viscous-turbulent effects strictly local
- admits parabolizing steady NS equations

Validation exercises for MLT closure

Asymptotic convergence theory confirmed

$$\left|e^h(n\Delta x)\right|_E \le Ch_e^{2\gamma}\left\|\text{data}\right\|_{L2}^2 + C_x\Delta x^3\left\|q_0\right\|_{H1}^2, \gamma = \min(k, r-1) \Rightarrow k$$

Validation exercises for TKE+low Re^t closure model

wall region meshing requirements are substantial
- algebraic non-linear Reynolds stress model
- transverse plane effects due to higher order phenomena